so, using $\cos^2\theta + \sin^2\theta = 1$, we have

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\,\sin\theta + r^2\sin^2\theta$$

$$+ \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\,\cos\theta + r^2\cos^2\theta$$

$$= \left(\frac{dr}{d\theta}\right)^2 + r^2$$

Assuming that $f'$ is continuous, we can use Theorem 10.2.5 to write the arc length as

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Therefore the length of a curve with polar equation $r = f(\theta)$, $a \leqslant \theta \leqslant b$, is

**5** $$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

**V EXAMPLE 4** Find the length of the cardioid $r = 1 + \sin\theta$.

SOLUTION The cardioid is shown in Figure 8. (We sketched it in Example 7 in Section 10.3.) Its full length is given by the parameter interval $0 \leqslant \theta \leqslant 2\pi$, so Formula 5 gives

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{(1 + \sin\theta)^2 + \cos^2\theta}\; d\theta$$

$$= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\; d\theta$$

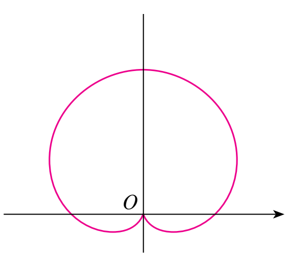



**FIGURE 8**
$r = 1 + \sin\theta$

We could evaluate this integral by multiplying and dividing the integrand by $\sqrt{2 - 2\sin\theta}$, or we could use a computer algebra system. In any event, we find that the length of the cardioid is $L = 8$.

## 10.4 Exercises

**1–4** Find the area of the region that is bounded by the given curve and lies in the specified sector.

**1.** $r = e^{-\theta/4}$, $\pi/2 \leqslant \theta \leqslant \pi$

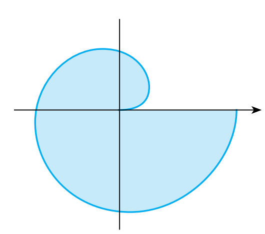
**2.** $r = \cos\theta$, $0 \leqslant \theta \leqslant \pi/6$

**3.** $r^2 = 9\sin 2\theta$, $r \geqslant 0$, $0 \leqslant \theta \leqslant \pi/2$

**4.** $r = \tan\theta$, $\pi/6 \leqslant \theta \leqslant \pi/3$

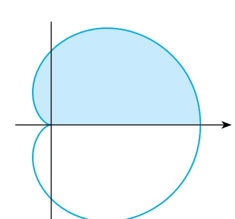
**5–8** Find the area of the shaded region.

**5.**



$r = \sqrt{\theta}$

**6.**

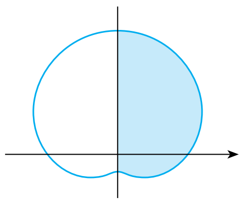


$r = 1 + \cos\theta$

Graphing calculator or computer required

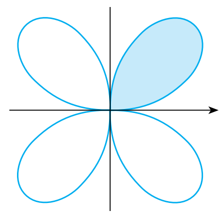
**1.** Homework Hints available at stewartcalculus.com

**7.**



$r = 4 + 3\sin\theta$

**8.**



$r = \sin 2\theta$

**9–12** Sketch the curve and find the area that it encloses.

**9.** $r = 2\sin\theta$

**10.** $r = 1 - \sin\theta$

**11.** $r = 3 + 2\cos\theta$

**12.** $r = 4 + 3\sin\theta$

**13–16** Graph the curve and find the area that it encloses.

**13.** $r = 2 + \sin 4\theta$

**14.** $r = 3 - 2\cos 4\theta$

**15.** $r = \sqrt{1 + \cos^2(5\theta)}$

**16.** $r = 1 + 5\sin 6\theta$

**17–21** Find the area of the region enclosed by one loop of the curve.

**17.** $r = 4\cos 3\theta$

**18.** $r^2 = \sin 2\theta$

**19.** $r = \sin 4\theta$

**20.** $r = 2\sin 5\theta$

**21.** $r = 1 + 2\sin\theta$ (inner loop)

**22.** Find the area enclosed by the loop of the **strophoid** $r = 2\cos\theta - \sec\theta$.

**23–28** Find the area of the region that lies inside the first curve and outside the second curve.

**23.** $r = 2\cos\theta,\quad r = 1$

**24.** $r = 1 - \sin\theta,\quad r = 1$

**25.** $r^2 = 8\cos 2\theta,\quad r = 2$

**26.** $r = 2 + \sin\theta,\quad r = 3\sin\theta$

**27.** $r = 3\cos\theta,\quad r = 1 + \cos\theta$

**28.** $r = 3\sin\theta,\quad r = 2 - \sin\theta$

**29–34** Find the area of the region that lies inside both curves.

**29.** $r = \sqrt{3}\cos\theta,\quad r = \sin\theta$

**30.** $r = 1 + \cos\theta,\quad r = 1 - \cos\theta$

**31.** $r = \sin 2\theta,\quad r = \cos 2\theta$

**32.** $r = 3 + 2\cos\theta,\quad r = 3 + 2\sin\theta$

**33.** $r^2 = \sin 2\theta,\quad r^2 = \cos 2\theta$

**34.** $r = a\sin\theta,\quad r = b\cos\theta,\quad a > 0,\ b > 0$

**35.** Find the area inside the larger loop and outside the smaller loop of the limaçon $r = \frac{1}{2} + \cos\theta$.

**36.** Find the area between a large loop and the enclosed small loop of the curve $r = 1 + 2\cos 3\theta$.

**37–42** Find all points of intersection of the given curves.

**37.** $r = 1 + \sin\theta,\quad r = 3\sin\theta$

**38.** $r = 1 - \cos\theta,\quad r = 1 + \sin\theta$

**39.** $r = 2\sin 2\theta,\quad r = 1$

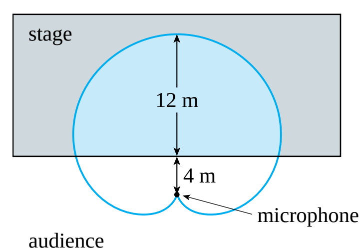
**40.** $r = \cos 3\theta,\quad r = \sin 3\theta$

**41.** $r = \sin\theta,\quad r = \sin 2\theta$

**42.** $r^2 = \sin 2\theta,\quad r^2 = \cos 2\theta$

**43.** The points of intersection of the cardioid $r = 1 + \sin\theta$ and the spiral loop $r = 2\theta$, $-\pi/2 \leqslant \theta \leqslant \pi/2$, can't be found exactly. Use a graphing device to find the approximate values of $\theta$ at which they intersect. Then use these values to estimate the area that lies inside both curves.

**44.** When recording live performances, sound engineers often use a microphone with a cardioid pickup pattern because it suppresses noise from the audience. Suppose the microphone is placed 4 m from the front of the stage (as in the figure) and the boundary of the optimal pickup region is given by the cardioid $r = 8 + 8\sin\theta$, where $r$ is measured in meters and the microphone is at the pole. The musicians want to know the area they will have on stage within the optimal pickup range of the microphone. Answer their question.




**45–48** Find the exact length of the polar curve.

**45.** $r = 2\cos\theta,\quad 0 \leqslant \theta \leqslant \pi$

**46.** $r = 5^\theta,\quad 0 \leqslant \theta \leqslant 2\pi$

**47.** $r = \theta^2,\quad 0 \leqslant \theta \leqslant 2\pi$

**48.** $r = 2(1 + \cos\theta)$

**49–50** Find the exact length of the curve. Use a graph to determine the parameter interval.

**49.** $r = \cos^4(\theta/4)$

**50.** $r = \cos^2(\theta/2)$

**51–54** Use a calculator to find the length of the curve correct to four decimal places. If necessary, graph the curve to determine the parameter interval.

**51.** One loop of the curve $r = \cos 2\theta$

**52.** $r = \tan\theta, \quad \pi/6 \leqslant \theta \leqslant \pi/3$

**53.** $r = \sin(6 \sin\theta)$

**54.** $r = \sin(\theta/4)$

**55.** (a) Use Formula 10.2.6 to show that the area of the surface generated by rotating the polar curve

$$r = f(\theta) \qquad a \leqslant \theta \leqslant b$$

(where $f'$ is continuous and $0 \leqslant a < b \leqslant \pi$) about the polar axis is

$$S = \int_a^b 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

(b) Use the formula in part (a) to find the surface area generated by rotating the lemniscate $r^2 = \cos 2\theta$ about the polar axis.

**56.** (a) Find a formula for the area of the surface generated by rotating the polar curve $r = f(\theta)$, $a \leqslant \theta \leqslant b$ (where $f'$ is continuous and $0 \leqslant a < b \leqslant \pi$), about the line $\theta = \pi/2$.

(b) Find the surface area generated by rotating the lemniscate $r^2 = \cos 2\theta$ about the line $\theta = \pi/2$.

## 10.5 Conic Sections

In this section we give geometric definitions of parabolas, ellipses, and hyperbolas and derive their standard equations. They are called **conic sections**, or **conics**, because they result from intersecting a cone with a plane as shown in Figure 1.

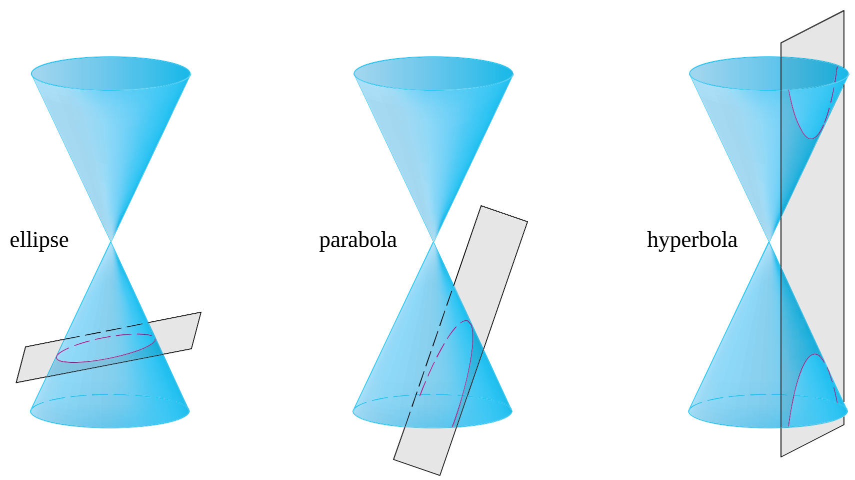



**FIGURE 1**
Conics

### Parabolas

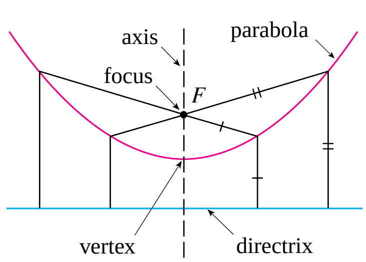



**FIGURE 2**

A **parabola** is the set of points in a plane that are equidistant from a fixed point $F$ (called the **focus**) and a fixed line (called the **directrix**). This definition is illustrated by Figure 2. Notice that the point halfway between the focus and the directrix lies on the parabola; it is called the **vertex**. The line through the focus perpendicular to the directrix is called the **axis** of the parabola.

In the 16th century Galileo showed that the path of a projectile that is shot into the air at an angle to the ground is a parabola. Since then, parabolic shapes have been used in designing automobile headlights, reflecting telescopes, and suspension bridges. (See Problem 20 on page 271 for the reflection property of parabolas that makes them so useful.)

We obtain a particularly simple equation for a parabola if we place its vertex at the origin $O$ and its directrix parallel to the $x$-axis as in Figure 3. If the focus is the point $(0, p)$, then the directrix has the equation $y = -p$. If $P(x, y)$ is any point on the parabola,