

FIGURE 19
 $\lim_{x \rightarrow \infty} f(x) = \infty$

Finally we note that an infinite limit at infinity can be defined as follows. The geometric illustration is given in Figure 19.

9 Definition Let f be a function defined on some interval (a, ∞) . Then

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

means that for every positive number M there is a corresponding positive number N such that

$$\text{if } x > N \quad \text{then} \quad f(x) > M$$

Similar definitions apply when the symbol ∞ is replaced by $-\infty$. (See Exercise 74.)

2.6 Exercises

1. Explain in your own words the meaning of each of the following.

(a) $\lim_{x \rightarrow \infty} f(x) = 5$ (b) $\lim_{x \rightarrow -\infty} f(x) = 3$

2. (a) Can the graph of $y = f(x)$ intersect a vertical asymptote? Can it intersect a horizontal asymptote? Illustrate by sketching graphs.

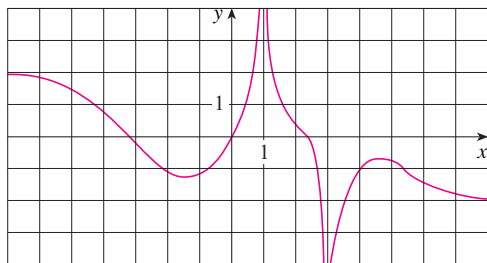
(b) How many horizontal asymptotes can the graph of $y = f(x)$ have? Sketch graphs to illustrate the possibilities.

3. For the function f whose graph is given, state the following.

(a) $\lim_{x \rightarrow \infty} f(x)$ (b) $\lim_{x \rightarrow -\infty} f(x)$

(c) $\lim_{x \rightarrow 1} f(x)$ (d) $\lim_{x \rightarrow 3} f(x)$

(e) The equations of the asymptotes



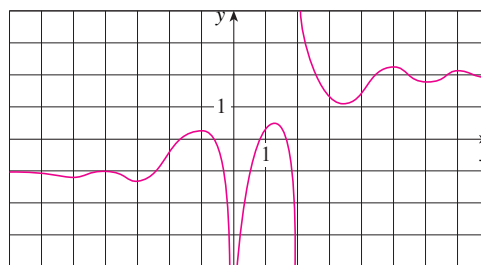
4. For the function g whose graph is given, state the following.

(a) $\lim_{x \rightarrow \infty} g(x)$ (b) $\lim_{x \rightarrow -\infty} g(x)$

(c) $\lim_{x \rightarrow 0} g(x)$ (d) $\lim_{x \rightarrow 2^-} g(x)$

(e) $\lim_{x \rightarrow 2^+} g(x)$

(f) The equations of the asymptotes



5–10 Sketch the graph of an example of a function f that satisfies all of the given conditions.

5. $\lim_{x \rightarrow 0} f(x) = -\infty$, $\lim_{x \rightarrow -\infty} f(x) = 5$, $\lim_{x \rightarrow \infty} f(x) = -5$

6. $\lim_{x \rightarrow 2} f(x) = \infty$, $\lim_{x \rightarrow -2^+} f(x) = \infty$, $\lim_{x \rightarrow -2^-} f(x) = -\infty$,
 $\lim_{x \rightarrow -\infty} f(x) = 0$, $\lim_{x \rightarrow \infty} f(x) = 0$, $f(0) = 0$

7. $\lim_{x \rightarrow 2} f(x) = -\infty$, $\lim_{x \rightarrow \infty} f(x) = \infty$, $\lim_{x \rightarrow -\infty} f(x) = 0$,
 $\lim_{x \rightarrow 0^+} f(x) = \infty$, $\lim_{x \rightarrow 0^-} f(x) = -\infty$

8. $\lim_{x \rightarrow \infty} f(x) = 3$, $\lim_{x \rightarrow 2^-} f(x) = \infty$, $\lim_{x \rightarrow 2^+} f(x) = -\infty$, f is odd

9. $f(0) = 3$, $\lim_{x \rightarrow 0^-} f(x) = 4$, $\lim_{x \rightarrow 0^+} f(x) = 2$,
 $\lim_{x \rightarrow -\infty} f(x) = -\infty$, $\lim_{x \rightarrow 4^-} f(x) = -\infty$, $\lim_{x \rightarrow 4^+} f(x) = \infty$,
 $\lim_{x \rightarrow \infty} f(x) = 3$

10. $\lim_{x \rightarrow 3} f(x) = -\infty$, $\lim_{x \rightarrow \infty} f(x) = 2$, $f(0) = 0$, f is even

11. Guess the value of the limit

$$\lim_{x \rightarrow \infty} \frac{x^2}{2^x}$$

by evaluating the function $f(x) = x^2/2^x$ for $x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 50,$ and 100 . Then use a graph of f to support your guess.

12. (a) Use a graph of

$$f(x) = \left(1 - \frac{2}{x}\right)^x$$

to estimate the value of $\lim_{x \rightarrow \infty} f(x)$ correct to two decimal places.

- (b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

13–14 Evaluate the limit and justify each step by indicating the appropriate properties of limits.

13. $\lim_{x \rightarrow \infty} \frac{3x^2 - x + 4}{2x^2 + 5x - 8}$

14. $\lim_{x \rightarrow \infty} \sqrt{\frac{12x^3 - 5x + 2}{1 + 4x^2 + 3x^3}}$

15–38 Find the limit or show that it does not exist.

15. $\lim_{x \rightarrow \infty} \frac{3x - 2}{2x + 1}$

16. $\lim_{x \rightarrow \infty} \frac{1 - x^2}{x^3 - x + 1}$

17. $\lim_{x \rightarrow \infty} \frac{x - 2}{x^2 + 1}$

18. $\lim_{x \rightarrow \infty} \frac{4x^3 + 6x^2 - 2}{2x^3 - 4x + 5}$

19. $\lim_{t \rightarrow \infty} \frac{\sqrt{t} + t^2}{2t - t^2}$

20. $\lim_{t \rightarrow \infty} \frac{t - t\sqrt{t}}{2t^{3/2} + 3t - 5}$

21. $\lim_{x \rightarrow \infty} \frac{(2x^2 + 1)^2}{(x - 1)^2(x^2 + x)}$

22. $\lim_{x \rightarrow \infty} \frac{x^2}{\sqrt{x^4 + 1}}$

23. $\lim_{x \rightarrow \infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1}$

24. $\lim_{x \rightarrow \infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1}$

25. $\lim_{x \rightarrow \infty} (\sqrt{9x^2 + x} - 3x)$

26. $\lim_{x \rightarrow \infty} (x + \sqrt{x^2 + 2x})$

27. $\lim_{x \rightarrow \infty} (\sqrt{x^2 + ax} - \sqrt{x^2 + bx})$

28. $\lim_{x \rightarrow \infty} \sqrt{x^2 + 1}$

29. $\lim_{x \rightarrow \infty} \frac{x^4 - 3x^2 + x}{x^3 - x + 2}$

30. $\lim_{x \rightarrow \infty} (e^{-x} + 2 \cos 3x)$

31. $\lim_{x \rightarrow \infty} (x^4 + x^5)$

32. $\lim_{x \rightarrow \infty} \frac{1 + x^6}{x^4 + 1}$

33. $\lim_{x \rightarrow \infty} \arctan(e^x)$

34. $\lim_{x \rightarrow \infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

35. $\lim_{x \rightarrow \infty} \frac{1 - e^x}{1 + 2e^x}$

36. $\lim_{x \rightarrow \infty} \frac{\sin^2 x}{x^2 + 1}$

37. $\lim_{x \rightarrow \infty} (e^{-2x} \cos x)$

38. $\lim_{x \rightarrow 0^+} \tan^{-1}(\ln x)$

39. (a) Estimate the value of

$$\lim_{x \rightarrow -\infty} (\sqrt{x^2 + x + 1} + x)$$

- by graphing the function $f(x) = \sqrt{x^2 + x + 1} + x$.
 (b) Use a table of values of $f(x)$ to guess the value of the limit.
 (c) Prove that your guess is correct.

40. (a) Use a graph of

$$f(x) = \sqrt{3x^2 + 8x + 6} - \sqrt{3x^2 + 3x + 1}$$

- to estimate the value of $\lim_{x \rightarrow \infty} f(x)$ to one decimal place.
 (b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.
 (c) Find the exact value of the limit.

41–46 Find the horizontal and vertical asymptotes of each curve. If you have a graphing device, check your work by graphing the curve and estimating the asymptotes.

41. $y = \frac{2x + 1}{x - 2}$

42. $y = \frac{x^2 + 1}{2x^2 - 3x - 2}$

43. $y = \frac{2x^2 + x - 1}{x^2 + x - 2}$

44. $y = \frac{1 + x^4}{x^2 - x^4}$

45. $y = \frac{x^3 - x}{x^2 - 6x + 5}$

46. $y = \frac{2e^x}{e^x - 5}$

47. Estimate the horizontal asymptote of the function

$$f(x) = \frac{3x^3 + 500x^2}{x^3 + 500x^2 + 100x + 2000}$$

by graphing f for $-10 \leq x \leq 10$. Then calculate the equation of the asymptote by evaluating the limit. How do you explain the discrepancy?

48. (a) Graph the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

How many horizontal and vertical asymptotes do you observe? Use the graph to estimate the values of the limits

$$\lim_{x \rightarrow \infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

- (b) By calculating values of $f(x)$, give numerical estimates of the limits in part (a).
 (c) Calculate the exact values of the limits in part (a). Did you get the same value or different values for these two limits? [In view of your answer to part (a), you might have to check your calculation for the second limit.]

69. (a) How large do we have to take x so that $1/x^2 < 0.0001$?
 (b) Taking $r = 2$ in Theorem 5, we have the statement

$$\lim_{x \rightarrow \infty} \frac{1}{x^2} = 0$$

Prove this directly using Definition 7.

70. (a) How large do we have to take x so that $1/\sqrt{x} < 0.0001$?
 (b) Taking $r = \frac{1}{2}$ in Theorem 5, we have the statement

$$\lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}} = 0$$

Prove this directly using Definition 7.

71. Use Definition 8 to prove that $\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$.

72. Prove, using Definition 9, that $\lim_{x \rightarrow \infty} x^3 = \infty$.

73. Use Definition 9 to prove that $\lim_{x \rightarrow \infty} e^x = \infty$.

74. Formulate a precise definition of

$$\lim_{x \rightarrow -\infty} f(x) = -\infty$$

Then use your definition to prove that

$$\lim_{x \rightarrow -\infty} (1 + x^3) = -\infty$$

75. Prove that

$$\lim_{x \rightarrow \infty} f(x) = \lim_{t \rightarrow 0^+} f(1/t)$$

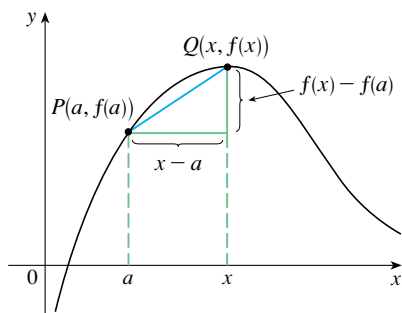
and

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{t \rightarrow 0^-} f(1/t)$$

if these limits exist.

2.7 Derivatives and Rates of Change

The problem of finding the tangent line to a curve and the problem of finding the velocity of an object both involve finding the same type of limit, as we saw in Section 2.1. This special type of limit is called a *derivative* and we will see that it can be interpreted as a rate of change in any of the sciences or engineering.



Tangents

If a curve C has equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ :

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Then we let Q approach P along the curve C by letting x approach a . If m_{PQ} approaches a number m , then we define the *tangent* t to be the line through P with slope m . (This amounts to saying that the tangent line is the limiting position of the secant line PQ as Q approaches P . See Figure 1.)

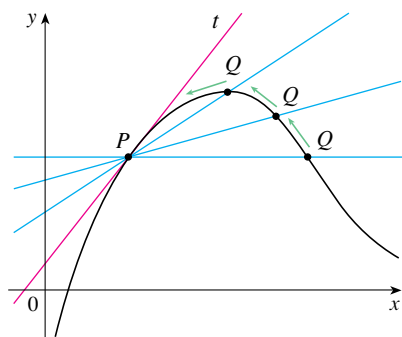


FIGURE 1

1 Definition The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

In our first example we confirm the guess we made in Example 1 in Section 2.1.