3.5 Exercises

1–4
(a) Find y' by implicit differentiation.
(b) Solve the equation explicitly for y and differentiate to get y' in terms of x.
(c) Check that your solutions to parts (a) and (b) are consistent by substituting the expression for y into your solution for part (a).

1. $9x^2 - y^2 = 1$

2. $2x^2 + x + xy = 1$

3. $\dfrac{1}{x} + \dfrac{1}{y} = 1$

4. $\cos x + \sqrt{y} = 5$

5–20 Find dy/dx by implicit differentiation.

5. $x^3 + y^3 = 1$

6. $2\sqrt{x} + \sqrt{y} = 3$

7. $x^2 + xy - y^2 = 4$

8. $2x^3 + x^2y - xy^3 = 2$

9. $x^4(x + y) = y^2(3x - y)$

10. $xe^y = x - y$

11. $y \cos x = x^2 + y^2$

12. $\cos(xy) = 1 + \sin y$

13. $4 \cos x \sin y = 1$

14. $e^y \sin x = x + xy$

15. $e^{x/y} = x - y$

16. $\sqrt{x + y} = 1 + x^2y^2$

17. $\tan^{-1}(x^2y) = x + xy^2$

18. $x \sin y + y \sin x = 1$

19. $e^y \cos x = 1 + \sin(xy)$

20. $\tan(x - y) = \dfrac{y}{1 + x^2}$

21. If $f(x) + x^2[f(x)]^3 = 10$ and $f(1) = 2$, find $f'(1)$.

22. If $g(x) + x \sin g(x) = x^2$, find $g'(0)$.

23–24 Regard y as the independent variable and x as the dependent variable and use implicit differentiation to find dx/dy.

23. $x^4y^2 - x^3y + 2xy^3 = 0$

24. $y \sec x = x \tan y$

25–32 Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

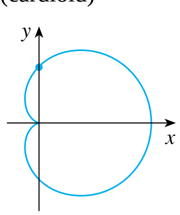
25. $y \sin 2x = x \cos 2y$, $(\pi/2, \pi/4)$

26. $\sin(x + y) = 2x - 2y$, (π, π)

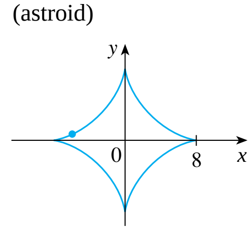
27. $x^2 + xy + y^2 = 3$, $(1, 1)$ (ellipse)

28. $x^2 + 2xy - y^2 + x = 2$, $(1, 2)$ (hyperbola)

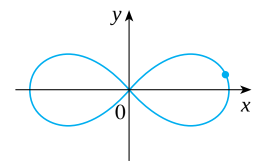
29. $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$
$(0, \frac{1}{2})$
(cardioid)



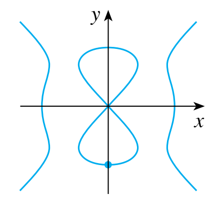
30. $x^{2/3} + y^{2/3} = 4$
$(-3\sqrt{3}, 1)$
(astroid)



31. $2(x^2 + y^2)^2 = 25(x^2 - y^2)$
$(3, 1)$
(lemniscate)



32. $y^2(y^2 - 4) = x^2(x^2 - 5)$
$(0, -2)$
(devil's curve)



33. (a) The curve with equation $y^2 = 5x^4 - x^2$ is called a **kampyle of Eudoxus**. Find an equation of the tangent line to this curve at the point $(1, 2)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on a common screen. (If your graphing device will graph implicitly defined curves, then use that capability. If not, you can still graph this curve by graphing its upper and lower halves separately.)

34. (a) The curve with equation $y^2 = x^3 + 3x^2$ is called the **Tschirnhausen cubic**. Find an equation of the tangent line to this curve at the point $(1, -2)$.
(b) At what points does this curve have horizontal tangents?
(c) Illustrate parts (a) and (b) by graphing the curve and the tangent lines on a common screen.

35–38 Find y'' by implicit differentiation.

35. $9x^2 + y^2 = 9$

36. $\sqrt{x} + \sqrt{y} = 1$

37. $x^3 + y^3 = 1$

38. $x^4 + y^4 = a^4$

39. If $xy + e^y = e$, find the value of y'' at the point where $x = 0$.

40. If $x^2 + xy + y^3 = 1$, find the value of y''' at the point where $x = 1$.

CAS **41.** Fanciful shapes can be created by using the implicit plotting capabilities of computer algebra systems.
(a) Graph the curve with equation

$$y(y^2 - 1)(y - 2) = x(x - 1)(x - 2)$$

At how many points does this curve have horizontal tangents? Estimate the x-coordinates of these points.
(b) Find equations of the tangent lines at the points $(0, 1)$ and $(0, 2)$.
(c) Find the exact x-coordinates of the points in part (a).
(d) Create even more fanciful curves by modifying the equation in part (a).

CAS **42.** (a) The curve with equation

$$2y^3 + y^2 - y^5 = x^4 - 2x^3 + x^2$$

has been likened to a bouncing wagon. Use a computer algebra system to graph this curve and discover why.

(b) At how many points does this curve have horizontal tangent lines? Find the x-coordinates of these points.

43. Find the points on the lemniscate in Exercise 31 where the tangent is horizontal.

44. Show by implicit differentiation that the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) is

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$$

45. Find an equation of the tangent line to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0).

46. Show that the sum of the x- and y-intercepts of any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is equal to c.

47. Show, using implicit differentiation, that any tangent line at a point P to a circle with center O is perpendicular to the radius OP.

48. The Power Rule can be proved using implicit differentiation for the case where n is a rational number, $n = p/q$, and $y = f(x) = x^n$ is assumed beforehand to be a differentiable function. If $y = x^{p/q}$, then $y^q = x^p$. Use implicit differentiation to show that

$$y' = \frac{p}{q} x^{(p/q)-1}$$

49–60 Find the derivative of the function. Simplify where possible.

49. $y = (\tan^{-1}x)^2$

50. $y = \tan^{-1}(x^2)$

51. $y = \sin^{-1}(2x + 1)$

52. $g(x) = \sqrt{x^2 - 1}\,\sec^{-1}x$

53. $G(x) = \sqrt{1 - x^2}\,\arccos x$

54. $y = \tan^{-1}\left(x - \sqrt{1 + x^2}\right)$

55. $h(t) = \cot^{-1}(t) + \cot^{-1}(1/t)$

56. $F(\theta) = \arcsin\sqrt{\sin\theta}$

57. $y = x\sin^{-1}x + \sqrt{1 - x^2}$

58. $y = \cos^{-1}(\sin^{-1}t)$

59. $y = \arccos\left(\dfrac{b + a\cos x}{a + b\cos x}\right), \quad 0 \leqslant x \leqslant \pi, \; a > b > 0$

60. $y = \arctan\sqrt{\dfrac{1 - x}{1 + x}}$

61–62 Find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

61. $f(x) = \sqrt{1 - x^2}\,\arcsin x$

62. $f(x) = \arctan(x^2 - x)$

63. Prove the formula for $(d/dx)(\cos^{-1}x)$ by the same method as for $(d/dx)(\sin^{-1}x)$.

64. (a) One way of defining $\sec^{-1}x$ is to say that $y = \sec^{-1}x \iff \sec y = x$ and $0 \leqslant y < \pi/2$ or $\pi \leqslant y < 3\pi/2$. Show that, with this definition,

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2 - 1}}$$

(b) Another way of defining $\sec^{-1}x$ that is sometimes used is to say that $y = \sec^{-1}x \iff \sec y = x$ and $0 \leqslant y \leqslant \pi$, $y \neq 0$. Show that, with this definition,

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2 - 1}}$$

65–68 Two curves are **orthogonal** if their tangent lines are perpendicular at each point of intersection. Show that the given families of curves are **orthogonal trajectories** of each other; that is, every curve in one family is orthogonal to every curve in the other family. Sketch both families of curves on the same axes.

65. $x^2 + y^2 = r^2, \quad ax + by = 0$

66. $x^2 + y^2 = ax, \quad x^2 + y^2 = by$

67. $y = cx^2, \quad x^2 + 2y^2 = k$

68. $y = ax^3, \quad x^2 + 3y^2 = b$

69. Show that the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the hyperbola $x^2/A^2 - y^2/B^2 = 1$ are orthogonal trajectories if $A^2 < a^2$ and $a^2 - b^2 = A^2 + B^2$ (so the ellipse and hyperbola have the same foci).

70. Find the value of the number a such that the families of curves $y = (x + c)^{-1}$ and $y = a(x + k)^{1/3}$ are orthogonal trajectories.

71. (a) The *van der Waals equation* for n moles of a gas is

$$\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT$$

where P is the pressure, V is the volume, and T is the temperature of the gas. The constant R is the universal gas constant and a and b are positive constants that are characteristic of a particular gas. If T remains constant, use implicit differentiation to find dV/dP.

(b) Find the rate of change of volume with respect to pressure of 1 mole of carbon dioxide at a volume of $V = 10$ L and a pressure of $P = 2.5$ atm. Use $a = 3.592$ $\text{L}^2\text{-atm/mole}^2$ and $b = 0.04267$ L/mole.

72. (a) Use implicit differentiation to find y' if $x^2 + xy + y^2 + 1 = 0$.

CAS (b) Plot the curve in part (a). What do you see? Prove that what you see is correct.

(c) In view of part (b), what can you say about the expression for y' that you found in part (a)?

73. The equation $x^2 - xy + y^2 = 3$ represents a "rotated ellipse," that is, an ellipse whose axes are not parallel to the coordinate axes. Find the points at which this ellipse crosses the x-axis and show that the tangent lines at these points are parallel.

74. (a) Where does the normal line to the ellipse $x^2 - xy + y^2 = 3$ at the point $(-1, 1)$ intersect the ellipse a second time?

(b) Illustrate part (a) by graphing the ellipse and the normal line.

75. Find all points on the curve $x^2y^2 + xy = 2$ where the slope of the tangent line is -1.

76. Find equations of both the tangent lines to the ellipse $x^2 + 4y^2 = 36$ that pass through the point $(12, 3)$.

77. (a) Suppose f is a one-to-one differentiable function and its inverse function f^{-1} is also differentiable. Use implicit differentiation to show that

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

provided that the denominator is not 0.

(b) If $f(4) = 5$ and $f'(4) = \frac{2}{3}$, find $(f^{-1})'(5)$.

78. (a) Show that $f(x) = x + e^x$ is one-to-one.

(b) What is the value of $f^{-1}(1)$?

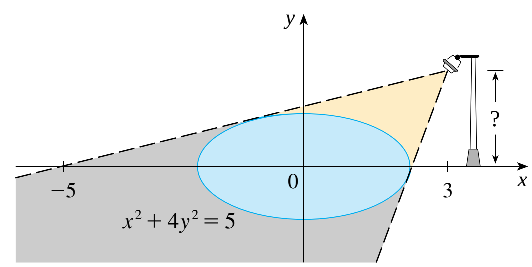
(c) Use the formula from Exercise 77(a) to find $(f^{-1})'(1)$.

79. The **Bessel function** of order 0, $y = J(x)$, satisfies the differential equation $xy'' + y' + xy = 0$ for all values of x and its value at 0 is $J(0) = 1$.

(a) Find $J'(0)$.

(b) Use implicit differentiation to find $J''(0)$.

80. The figure shows a lamp located three units to the right of the y-axis and a shadow created by the elliptical region $x^2 + 4y^2 \leq 5$. If the point $(-5, 0)$ is on the edge of the shadow, how far above the x-axis is the lamp located?



LABORATORY PROJECT CAS FAMILIES OF IMPLICIT CURVES

In this project you will explore the changing shapes of implicitly defined curves as you vary the constants in a family, and determine which features are common to all members of the family.

1. Consider the family of curves

$$y^2 - 2x^2(x + 8) = c[(y + 1)^2(y + 9) - x^2]$$

(a) By graphing the curves with $c = 0$ and $c = 2$, determine how many points of intersection there are. (You might have to zoom in to find all of them.)

(b) Now add the curves with $c = 5$ and $c = 10$ to your graphs in part (a). What do you notice? What about other values of c?

2. (a) Graph several members of the family of curves

$$x^2 + y^2 + cx^2y^2 = 1$$

Describe how the graph changes as you change the value of c.

(b) What happens to the curve when $c = -1$? Describe what appears on the screen. Can you prove it algebraically?

(c) Find y' by implicit differentiation. For the case $c = -1$, is your expression for y' consistent with what you discovered in part (b)?

CAS Computer algebra system required