Exercises

1–6 Find the local and absolute extreme values of the function on the given interval.

1. $f(x) = x^3 - 6x^2 + 9x + 1, \quad [2, 4]$
2. $f(x) = x\sqrt{1 - x}, \quad [-1, 1]$
3. $f(x) = \dfrac{3x - 4}{x^2 + 1}, \quad [-2, 2]$
4. $f(x) = \sqrt{x^2 + x + 1}, \quad [-2, 1]$
5. $f(x) = x + 2\cos x, \quad [-\pi, \pi]$
6. $f(x) = x^2 e^{-x}, \quad [-1, 3]$

7–14 Evaluate the limit.

7. $\displaystyle\lim_{x \to 0} \frac{e^x - 1}{\tan x}$
8. $\displaystyle\lim_{x \to 0} \frac{\tan 4x}{x + \sin 2x}$
9. $\displaystyle\lim_{x \to 0} \frac{e^{4x} - 1 - 4x}{x^2}$
10. $\displaystyle\lim_{x \to \infty} \frac{e^{4x} - 1 - 4x}{x^2}$
11. $\displaystyle\lim_{x \to -\infty} (x^2 - x^3)e^{2x}$
12. $\displaystyle\lim_{x \to \pi^-} (x - \pi)\csc x$
13. $\displaystyle\lim_{x \to 1^+} \left(\frac{x}{x - 1} - \frac{1}{\ln x}\right)$
14. $\displaystyle\lim_{x \to (\pi/2)^-} (\tan x)^{\cos x}$

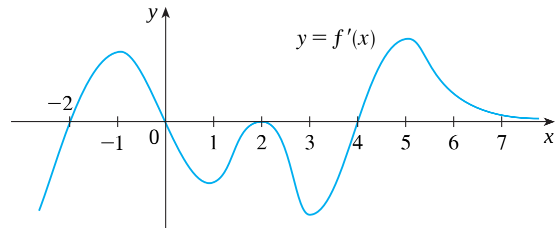
15–17 Sketch the graph of a function that satisfies the given conditions.

15. $f(0) = 0, \quad f'(-2) = f'(1) = f'(9) = 0,$
$\lim_{x \to \infty} f(x) = 0, \quad \lim_{x \to 6} f(x) = -\infty,$
$f'(x) < 0$ on $(-\infty, -2)$, $(1, 6)$, and $(9, \infty)$,
$f'(x) > 0$ on $(-2, 1)$ and $(6, 9)$,
$f''(x) > 0$ on $(-\infty, 0)$ and $(12, \infty)$,
$f''(x) < 0$ on $(0, 6)$ and $(6, 12)$

16. $f(0) = 0$, f is continuous and even,
$f'(x) = 2x$ if $0 < x < 1$, $f'(x) = -1$ if $1 < x < 3$,
$f'(x) = 1$ if $x > 3$

17. f is odd, $f'(x) < 0$ for $0 < x < 2$,
$f'(x) > 0$ for $x > 2$, $f''(x) > 0$ for $0 < x < 3$,
$f''(x) < 0$ for $x > 3$, $\lim_{x \to \infty} f(x) = -2$

18. The figure shows the graph of the *derivative* f' of a function f.
(a) On what intervals is f increasing or decreasing?
(b) For what values of x does f have a local maximum or minimum?
(c) Sketch the graph of f''.
(d) Sketch a possible graph of f.



19–34 Use the guidelines of Section 4.5 to sketch the curve.

19. $y = 2 - 2x - x^3$
20. $y = x^3 - 6x^2 - 15x + 4$
21. $y = x^4 - 3x^3 + 3x^2 - x$
22. $y = \dfrac{x}{1 - x^2}$
23. $y = \dfrac{1}{x(x - 3)^2}$
24. $y = \dfrac{1}{x^2} - \dfrac{1}{(x - 2)^2}$
25. $y = x^2/(x + 8)$
26. $y = \sqrt{1 - x} + \sqrt{1 + x}$
27. $y = x\sqrt{2 + x}$
28. $y = \sqrt[3]{x^2 + 1}$
29. $y = e^x \sin x, \quad -\pi \leqslant x \leqslant \pi$
30. $y = 4x - \tan x, \quad -\pi/2 < x < \pi/2$
31. $y = \sin^{-1}(1/x)$
32. $y = e^{2x - x^2}$
33. $y = (x - 2)e^{-x}$
34. $y = x + \ln(x^2 + 1)$

35–38 Produce graphs of f that reveal all the important aspects of the curve. Use graphs of f' and f'' to estimate the intervals of increase and decrease, extreme values, intervals of concavity, and inflection points. In Exercise 35 use calculus to find these quantities exactly.

35. $f(x) = \dfrac{x^2 - 1}{x^3}$
36. $f(x) = \dfrac{x^3 - x}{x^2 + x + 3}$
37. $f(x) = 3x^6 - 5x^5 + x^4 - 5x^3 - 2x^2 + 2$
38. $f(x) = x^2 + 6.5\sin x, \quad -5 \leqslant x \leqslant 5$

39. Graph $f(x) = e^{-1/x^2}$ in a viewing rectangle that shows all the main aspects of this function. Estimate the inflection points. Then use calculus to find them exactly.

CAS 40. (a) Graph the function $f(x) = 1/(1 + e^{1/x})$.
(b) Explain the shape of the graph by computing the limits of $f(x)$ as x approaches ∞, $-\infty$, 0^+, and 0^-.
(c) Use the graph of f to estimate the coordinates of the inflection points.
(d) Use your CAS to compute and graph f''.
(e) Use the graph in part (d) to estimate the inflection points more accurately.

Graphing calculator or computer required   CAS Computer algebra system required

CAS **41–42** Use the graphs of f, f', and f'' to estimate the x-coordinates of the maximum and minimum points and inflection points of f.

41. $f(x) = \dfrac{\cos^2 x}{\sqrt{x^2 + x + 1}}, \quad -\pi \leqslant x \leqslant \pi$

42. $f(x) = e^{-0.1x} \ln(x^2 - 1)$

43. Investigate the family of functions $f(x) = \ln(\sin x + C)$. What features do the members of this family have in common? How do they differ? For which values of C is f continuous on $(-\infty, \infty)$? For which values of C does f have no graph at all? What happens as $C \to \infty$?

44. Investigate the family of functions $f(x) = cxe^{-cx^2}$. What happens to the maximum and minimum points and the inflection points as c changes? Illustrate your conclusions by graphing several members of the family.

45. Show that the equation $3x + 2\cos x + 5 = 0$ has exactly one real root.

46. Suppose that f is continuous on $[0, 4]$, $f(0) = 1$, and $2 \leqslant f'(x) \leqslant 5$ for all x in $(0, 4)$. Show that $9 \leqslant f(4) \leqslant 21$.

47. By applying the Mean Value Theorem to the function $f(x) = x^{1/5}$ on the interval $[32, 33]$, show that

$$2 < \sqrt[5]{33} < 2.0125$$

48. For what values of the constants a and b is $(1, 3)$ a point of inflection of the curve $y = ax^3 + bx^2$?

49. Let $g(x) = f(x^2)$, where f is twice differentiable for all x, $f'(x) > 0$ for all $x \neq 0$, and f is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$.
(a) At what numbers does g have an extreme value?
(b) Discuss the concavity of g.

50. Find two positive integers such that the sum of the first number and four times the second number is 1000 and the product of the numbers is as large as possible.

51. Show that the shortest distance from the point (x_1, y_1) to the straight line $Ax + By + C = 0$ is

$$\frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

52. Find the point on the hyperbola $xy = 8$ that is closest to the point $(3, 0)$.

53. Find the smallest possible area of an isosceles triangle that is circumscribed about a circle of radius r.

54. Find the volume of the largest circular cone that can be inscribed in a sphere of radius r.

55. In ΔABC, D lies on AB, $CD \perp AB$, $|AD| = |BD| = 4$ cm, and $|CD| = 5$ cm. Where should a point P be chosen on CD so that the sum $|PA| + |PB| + |PC|$ is a minimum?

56. Solve Exercise 55 when $|CD| = 2$ cm.

57. The velocity of a wave of length L in deep water is

$$v = K\sqrt{\frac{L}{C} + \frac{C}{L}}$$

where K and C are known positive constants. What is the length of the wave that gives the minimum velocity?

58. A metal storage tank with volume V is to be constructed in the shape of a right circular cylinder surmounted by a hemisphere. What dimensions will require the least amount of metal?

59. A hockey team plays in an arena with a seating capacity of 15,000 spectators. With the ticket price set at \$12, average attendance at a game has been 11,000. A market survey indicates that for each dollar the ticket price is lowered, average attendance will increase by 1000. How should the owners of the team set the ticket price to maximize their revenue from ticket sales?

60. A manufacturer determines that the cost of making x units of a commodity is $C(x) = 1800 + 25x - 0.2x^2 + 0.001x^3$ and the demand function is $p(x) = 48.2 - 0.03x$.
(a) Graph the cost and revenue functions and use the graphs to estimate the production level for maximum profit.
(b) Use calculus to find the production level for maximum profit.
(c) Estimate the production level that minimizes the average cost.

61. Use Newton's method to find the root of the equation $x^5 - x^4 + 3x^2 - 3x - 2 = 0$ in the interval $[1, 2]$ correct to six decimal places.

62. Use Newton's method to find all roots of the equation $\sin x = x^2 - 3x + 1$ correct to six decimal places.

63. Use Newton's method to find the absolute maximum value of the function $f(t) = \cos t + t - t^2$ correct to eight decimal places.

64. Use the guidelines in Section 4.5 to sketch the curve $y = x \sin x$, $0 \leqslant x \leqslant 2\pi$. Use Newton's method when necessary.

65–72 Find f.

65. $f'(x) = \cos x - (1 - x^2)^{-1/2}$

66. $f'(x) = 2e^x + \sec x \tan x$

67. $f'(x) = \sqrt{x^3} + \sqrt[3]{x^2}$

68. $f'(x) = \sinh x + 2\cosh x, \quad f(0) = 2$

69. $f'(t) = 2t - 3\sin t, \quad f(0) = 5$

70. $f'(u) = \dfrac{u^2 + \sqrt{u}}{u}, \quad f(1) = 3$

71. $f''(x) = 1 - 6x + 48x^2, \quad f(0) = 1, \quad f'(0) = 2$

72. $f''(x) = 2x^3 + 3x^2 - 4x + 5, \quad f(0) = 2, \quad f(1) = 0$

73–74 A particle is moving with the given data. Find the position of the particle.

73. $v(t) = 2t - 1/(1 + t^2)$, $s(0) = 1$

74. $a(t) = \sin t + 3 \cos t$, $s(0) = 0$, $v(0) = 2$

75. (a) If $f(x) = 0.1e^x + \sin x$, $-4 \leqslant x \leqslant 4$, use a graph of f to sketch a rough graph of the antiderivative F of f that satisfies $F(0) = 0$.
(b) Find an expression for $F(x)$.
(c) Graph F using the expression in part (b). Compare with your sketch in part (a).

76. Investigate the family of curves given by

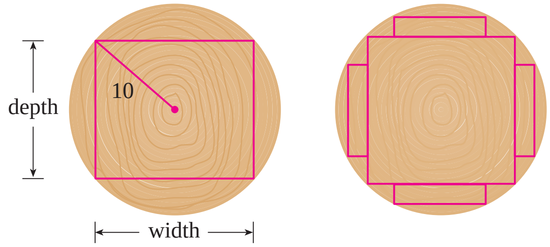
$$f(x) = x^4 + x^3 + cx^2$$

In particular you should determine the transitional value of c at which the number of critical numbers changes and the transitional value at which the number of inflection points changes. Illustrate the various possible shapes with graphs.

77. A canister is dropped from a helicopter 500 m above the ground. Its parachute does not open, but the canister has been designed to withstand an impact velocity of 100 m/s. Will it burst?

78. In an automobile race along a straight road, car A passed car B twice. Prove that at some time during the race their accelerations were equal. State the assumptions that you make.

79. A rectangular beam will be cut from a cylindrical log of radius 10 inches.
(a) Show that the beam of maximal cross-sectional area is a square.
(b) Four rectangular planks will be cut from the four sections of the log that remain after cutting the square beam. Determine the dimensions of the planks that will have maximal cross-sectional area.
(c) Suppose that the strength of a rectangular beam is proportional to the product of its width and the square of its depth. Find the dimensions of the strongest beam that can be cut from the cylindrical log.



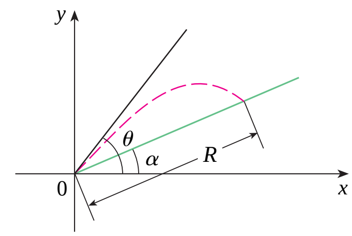
80. If a projectile is fired with an initial velocity v at an angle of inclination θ from the horizontal, then its trajectory, neglecting air resistance, is the parabola

$$y = (\tan\theta)x - \frac{g}{2v^2\cos^2\theta}x^2 \qquad 0 < \theta < \frac{\pi}{2}$$

(a) Suppose the projectile is fired from the base of a plane that is inclined at an angle α, $\alpha > 0$, from the horizontal, as shown in the figure. Show that the range of the projectile, measured up the slope, is given by

$$R(\theta) = \frac{2v^2\cos\theta\,\sin(\theta - \alpha)}{g\cos^2\alpha}$$

(b) Determine θ so that R is a maximum.
(c) Suppose the plane is at an angle α *below* the horizontal. Determine the range R in this case, and determine the angle at which the projectile should be fired to maximize R.

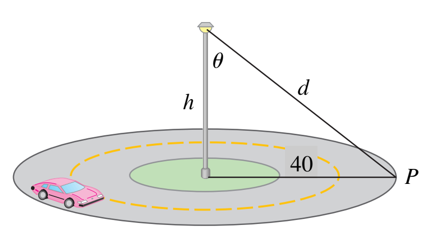


81. Show that, for $x > 0$,

$$\frac{x}{1 + x^2} < \tan^{-1}x < x$$

82. Sketch the graph of a function f such that $f'(x) < 0$ for all x, $f''(x) > 0$ for $|x| > 1$, $f''(x) < 0$ for $|x| < 1$, and $\lim_{x\to\pm\infty}[f(x) + x] = 0$.

83. A light is to be placed atop a pole of height h feet to illuminate a busy traffic circle, which has a radius of 40 ft. The intensity of illumination I at any point P on the circle is directly proportional to the cosine of the angle θ (see the figure) and inversely proportional to the square of the distance d from the source.
(a) How tall should the light pole be to maximize I?
(b) Suppose that the light pole is h feet tall and that a woman is walking away from the base of the pole at the rate of 4 ft/s. At what rate is the intensity of the light at the point on her back 4 ft above the ground decreasing when she reaches the outer edge of the traffic circle?



84. Water is flowing at a constant rate into a spherical tank. Let $V(t)$ be the volume of water in the tank and $H(t)$ be the height of the water in the tank at time t.
(a) What are the meanings of $V'(t)$ and $H'(t)$? Are these derivatives positive, negative, or zero?
(b) Is $V''(t)$ positive, negative, or zero? Explain.
(c) Let t_1, t_2, and t_3 be the times when the tank is one-quarter full, half full, and three-quarters full, respectively. Are the values $H''(t_1)$, $H''(t_2)$, and $H''(t_3)$ positive, negative, or zero? Why?