

Since $f'(x) = 0$ for all x , we have $f'(c) = 0$, and so Equation 6 becomes

$$f(x_2) - f(x_1) = 0 \quad \text{or} \quad f(x_2) = f(x_1)$$

Therefore f has the same value at *any* two numbers x_1 and x_2 in (a, b) . This means that f is constant on (a, b) .

7 Corollary If $f'(x) = g'(x)$ for all x in an interval (a, b) , then $f - g$ is constant on (a, b) ; that is, $f(x) = g(x) + c$ where c is a constant.

PROOF Let $F(x) = f(x) - g(x)$. Then

$$F'(x) = f'(x) - g'(x) = 0$$

for all x in (a, b) . Thus, by Theorem 5, F is constant; that is, $f - g$ is constant.

NOTE Care must be taken in applying Theorem 5. Let

$$f(x) = \frac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

The domain of f is $D = \{x \mid x \neq 0\}$ and $f'(x) = 0$ for all x in D . But f is obviously not a constant function. This does not contradict Theorem 5 because D is not an interval. Notice that f is constant on the interval $(0, \infty)$ and also on the interval $(-\infty, 0)$.

EXAMPLE 6 Prove the identity $\tan^{-1}x + \cot^{-1}x = \pi/2$.

SOLUTION Although calculus isn't needed to prove this identity, the proof using calculus is quite simple. If $f(x) = \tan^{-1}x + \cot^{-1}x$, then

$$f'(x) = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$$

for all values of x . Therefore $f(x) = C$, a constant. To determine the value of C , we put $x = 1$ [because we can evaluate $f(1)$ exactly]. Then

$$C = f(1) = \tan^{-1}1 + \cot^{-1}1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

Thus $\tan^{-1}x + \cot^{-1}x = \pi/2$.

4.2 Exercises

1–4 Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

1. $f(x) = 5 - 12x + 3x^2$, $[1, 3]$

2. $f(x) = x^3 - x^2 - 6x + 2$, $[0, 3]$

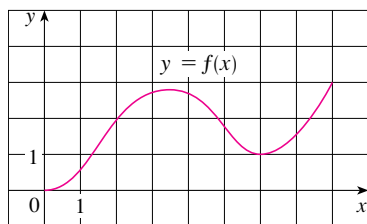
3. $f(x) = \sqrt{x} - \frac{1}{3}x$, $[0, 9]$

4. $f(x) = \cos 2x$, $[\pi/8, 7\pi/8]$

5. Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in $(-1, 1)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

6. Let $f(x) = \tan x$. Show that $f(0) = f(\pi)$ but there is no number c in $(0, \pi)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

7. Use the graph of f to estimate the values of c that satisfy the conclusion of the Mean Value Theorem for the interval $[0, 8]$.



8. Use the graph of f given in Exercise 7 to estimate the values of c that satisfy the conclusion of the Mean Value Theorem for the interval $[1, 7]$.


9–12 Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

9. $f(x) = 2x^2 - 3x + 1$, $[0, 2]$

10. $f(x) = x^3 - 3x + 2$, $[-2, 2]$

11. $f(x) = \ln x$, $[1, 4]$

12. $f(x) = 1/x$, $[1, 3]$

 **13–14** Find the number c that satisfies the conclusion of the Mean Value Theorem on the given interval. Graph the function, the secant line through the endpoints, and the tangent line at $(c, f(c))$. Are the secant line and the tangent line parallel?

13. $f(x) = \sqrt{x}$, $[0, 4]$

14. $f(x) = e^{-x}$, $[0, 2]$

15. Let $f(x) = (x - 3)^{-2}$. Show that there is no value of c in $(1, 4)$ such that $f(4) - f(1) = f'(c)(4 - 1)$. Why does this not contradict the Mean Value Theorem?

16. Let $f(x) = 2 - |2x - 1|$. Show that there is no value of c such that $f(3) - f(0) = f'(c)(3 - 0)$. Why does this not contradict the Mean Value Theorem?

17–18 Show that the equation has exactly one real root.

17. $2x + \cos x = 0$

18. $x^3 + e^x = 0$

19. Show that the equation $x^3 - 15x + c = 0$ has at most one root in the interval $[-2, 2]$.

20. Show that the equation $x^4 + 4x + c = 0$ has at most two real roots.

21. (a) Show that a polynomial of degree 3 has at most three real roots.

- (b) Show that a polynomial of degree n has at most n real roots.

22. (a) Suppose that f is differentiable on \mathbb{R} and has two roots. Show that f' has at least one root.

- (b) Suppose f is twice differentiable on \mathbb{R} and has three roots. Show that f'' has at least one real root.

- (c) Can you generalize parts (a) and (b)?

23. If $f(1) = 10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$, how small can $f(4)$ possibly be?

24. Suppose that $3 \leq f'(x) \leq 5$ for all values of x . Show that $18 \leq f(8) - f(2) \leq 30$.

25. Does there exist a function f such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leq 2$ for all x ?

26. Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b) . Suppose also that $f(a) = g(a)$ and $f'(x) < g'(x)$ for $a < x < b$. Prove that $f(b) < g(b)$. [Hint: Apply the Mean Value Theorem to the function $h = f - g$.]

27. Show that $\sqrt{1+x} < 1 + \frac{1}{2}x$ if $x > 0$.

28. Suppose f is an odd function and is differentiable everywhere. Prove that for every positive number b , there exists a number c in $(-b, b)$ such that $f'(c) = f(b)/b$.

29. Use the Mean Value Theorem to prove the inequality

$$|\sin a - \sin b| \leq |a - b| \quad \text{for all } a \text{ and } b$$

30. If $f'(x) = c$ (c a constant) for all x , use Corollary 7 to show that $f(x) = cx + d$ for some constant d .

31. Let $f(x) = 1/x$ and

$$g(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ 1 + \frac{1}{x} & \text{if } x < 0 \end{cases}$$

Show that $f'(x) = g'(x)$ for all x in their domains. Can we conclude from Corollary 7 that $f - g$ is constant?

32. Use the method of Example 6 to prove the identity

$$2 \sin^{-1}x = \cos^{-1}(1 - 2x^2) \quad x \geq 0$$

33. Prove the identity

$$\arcsin \frac{x-1}{x+1} = 2 \arctan \sqrt{x} - \frac{\pi}{2}$$

34. At 2:00 PM a car's speedometer reads 30 mi/h. At 2:10 PM it reads 50 mi/h. Show that at some time between 2:00 and 2:10 the acceleration is exactly 120 mi/h².

35. Two runners start a race at the same time and finish in a tie. Prove that at some time during the race they have the same speed. [Hint: Consider $f(t) = g(t) - h(t)$, where g and h are the position functions of the two runners.]

36. A number a is called a **fixed point** of a function f if $f(a) = a$. Prove that if $f'(x) \neq 1$ for all real numbers x , then f has at most one fixed point.