So far we have computed the limit of $\ln y$, but what we want is the limit of y. To find this we use the fact that $y = e^{\ln y}$:

$$\lim_{x\to 0^+} (1 + \sin 4x)^{\cot x} = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^4$$

The graph of the function $y = x^x$, $x > 0$, is shown in Figure 6. Notice that although 0^0 is not defined, the values of the function approach 1 as $x \to 0^+$. This confirms the result of Example 9.

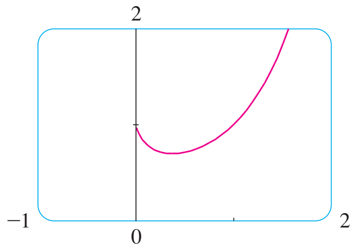


FIGURE 6

V EXAMPLE 9 Find $\lim_{x\to 0^+} x^x$.

SOLUTION Notice that this limit is indeterminate since $0^x = 0$ for any $x > 0$ but $x^0 = 1$ for any $x \neq 0$. We could proceed as in Example 8 or by writing the function as an exponential:

$$x^x = (e^{\ln x})^x = e^{x \ln x}$$

In Example 6 we used l'Hospital's Rule to show that

$$\lim_{x\to 0^+} x \ln x = 0$$

Therefore

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x \ln x} = e^0 = 1$$

4.4 Exercises

1–4 Given that

$$\lim_{x\to a} f(x) = 0 \qquad \lim_{x\to a} g(x) = 0 \qquad \lim_{x\to a} h(x) = 1$$
$$\lim_{x\to a} p(x) = \infty \qquad \lim_{x\to a} q(x) = \infty$$

which of the following limits are indeterminate forms? For those that are not an indeterminate form, evaluate the limit where possible.

1. (a) $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ (b) $\lim_{x\to a} \dfrac{f(x)}{p(x)}$ (c) $\lim_{x\to a} \dfrac{h(x)}{p(x)}$

(d) $\lim_{x\to a} \dfrac{p(x)}{f(x)}$ (e) $\lim_{x\to a} \dfrac{p(x)}{q(x)}$

2. (a) $\lim_{x\to a} [f(x)p(x)]$ (b) $\lim_{x\to a} [h(x)p(x)]$

(c) $\lim_{x\to a} [p(x)q(x)]$

3. (a) $\lim_{x\to a} [f(x) - p(x)]$ (b) $\lim_{x\to a} [p(x) - q(x)]$

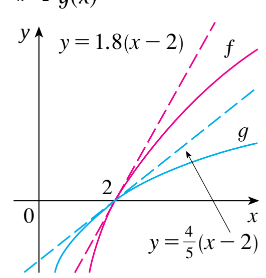
(c) $\lim_{x\to a} [p(x) + q(x)]$

4. (a) $\lim_{x\to a} [f(x)]^{g(x)}$ (b) $\lim_{x\to a} [f(x)]^{p(x)}$ (c) $\lim_{x\to a} [h(x)]^{p(x)}$

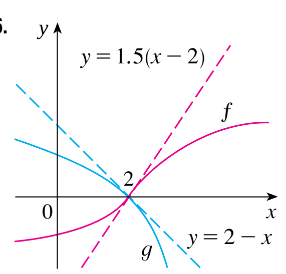
(d) $\lim_{x\to a} [p(x)]^{f(x)}$ (e) $\lim_{x\to a} [p(x)]^{q(x)}$ (f) $\lim_{x\to a} \sqrt[q(x)]{p(x)}$

5–6 Use the graphs of f and g and their tangent lines at (2, 0) to find $\lim_{x\to 2} \dfrac{f(x)}{g(x)}$.

5.



6.



7–66 Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule doesn't apply, explain why.

7. $\lim_{x\to 1} \dfrac{x^2 - 1}{x^2 - x}$

8. $\lim_{x\to 2} \dfrac{x^2 + x - 6}{x - 2}$

9. $\lim_{x\to 1} \dfrac{x^3 - 2x^2 + 1}{x^3 - 1}$

10. $\lim_{x\to 1/2} \dfrac{6x^2 + 5x - 4}{4x^2 + 16x - 9}$

11. $\lim_{x\to (\pi/2)^+} \dfrac{\cos x}{1 - \sin x}$

12. $\lim_{x\to 0} \dfrac{\sin 4x}{\tan 5x}$

13. $\lim_{t\to 0} \dfrac{e^{2t} - 1}{\sin t}$

14. $\lim_{x\to 0} \dfrac{x^2}{1 - \cos x}$

15. $\displaystyle\lim_{\theta\to\pi/2}\frac{1-\sin\theta}{1+\cos 2\theta}$

16. $\displaystyle\lim_{\theta\to\pi/2}\frac{1-\sin\theta}{\csc\theta}$

17. $\displaystyle\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}$

18. $\displaystyle\lim_{x\to\infty}\frac{x+x^2}{1-2x^2}$

19. $\displaystyle\lim_{x\to 0^+}\frac{\ln x}{x}$

20. $\displaystyle\lim_{x\to\infty}\frac{\ln\sqrt{x}}{x^2}$

21. $\displaystyle\lim_{t\to 1}\frac{t^8-1}{t^5-1}$

22. $\displaystyle\lim_{t\to 0}\frac{8^t-5^t}{t}$

23. $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+2x}-\sqrt{1-4x}}{x}$

24. $\displaystyle\lim_{u\to\infty}\frac{e^{u/10}}{u^3}$

25. $\displaystyle\lim_{x\to 0}\frac{e^x-1-x}{x^2}$

26. $\displaystyle\lim_{x\to 0}\frac{\sinh x-x}{x^3}$

27. $\displaystyle\lim_{x\to 0}\frac{\tanh x}{\tan x}$

28. $\displaystyle\lim_{x\to 0}\frac{x-\sin x}{x-\tan x}$

29. $\displaystyle\lim_{x\to 0}\frac{\sin^{-1}x}{x}$

30. $\displaystyle\lim_{x\to\infty}\frac{(\ln x)^2}{x}$

31. $\displaystyle\lim_{x\to 0}\frac{x3^x}{3^x-1}$

32. $\displaystyle\lim_{x\to 0}\frac{\cos mx-\cos nx}{x^2}$

33. $\displaystyle\lim_{x\to 0}\frac{x+\sin x}{x+\cos x}$

34. $\displaystyle\lim_{x\to 0}\frac{x}{\tan^{-1}(4x)}$

35. $\displaystyle\lim_{x\to 1}\frac{1-x+\ln x}{1+\cos\pi x}$

36. $\displaystyle\lim_{x\to 0^+}\frac{x^x-1}{\ln x+x-1}$

37. $\displaystyle\lim_{x\to 1}\frac{x^a-ax+a-1}{(x-1)^2}$

38. $\displaystyle\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x}$

39. $\displaystyle\lim_{x\to 0}\frac{\cos x-1+\frac{1}{2}x^2}{x^4}$

40. $\displaystyle\lim_{x\to a^+}\frac{\cos x\ln(x-a)}{\ln(e^x-e^a)}$

41. $\displaystyle\lim_{x\to\infty}x\sin(\pi/x)$

42. $\displaystyle\lim_{x\to\infty}\sqrt{x}\,e^{-x/2}$

43. $\displaystyle\lim_{x\to 0}\cot 2x\sin 6x$

44. $\displaystyle\lim_{x\to 0^+}\sin x\ln x$

45. $\displaystyle\lim_{x\to\infty}x^3e^{-x^2}$

46. $\displaystyle\lim_{x\to\infty}x\tan(1/x)$

47. $\displaystyle\lim_{x\to 1^+}\ln x\tan(\pi x/2)$

48. $\displaystyle\lim_{x\to(\pi/2)^-}\cos x\sec 5x$

49. $\displaystyle\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$

50. $\displaystyle\lim_{x\to 0}(\csc x-\cot x)$

51. $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$

52. $\displaystyle\lim_{x\to 0}\left(\cot x-\frac{1}{x}\right)$

53. $\displaystyle\lim_{x\to\infty}(x-\ln x)$

54. $\displaystyle\lim_{x\to 1^+}[\ln(x^7-1)-\ln(x^5-1)]$

55. $\displaystyle\lim_{x\to 0^+}x^{\sqrt{x}}$

56. $\displaystyle\lim_{x\to 0^+}(\tan 2x)^x$

57. $\displaystyle\lim_{x\to 0}(1-2x)^{1/x}$

58. $\displaystyle\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}$

59. $\displaystyle\lim_{x\to 1^+}x^{1/(1-x)}$

60. $\displaystyle\lim_{x\to\infty}x^{(\ln 2)/(1+\ln x)}$

61. $\displaystyle\lim_{x\to\infty}x^{1/x}$

62. $\displaystyle\lim_{x\to\infty}(e^x+x)^{1/x}$

63. $\displaystyle\lim_{x\to 0^+}(4x+1)^{\cot x}$

64. $\displaystyle\lim_{x\to 1}(2-x)^{\tan(\pi x/2)}$

65. $\displaystyle\lim_{x\to 0^+}(\cos x)^{1/x^2}$

66. $\displaystyle\lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1}$

67–68 Use a graph to estimate the value of the limit. Then use l'Hospital's Rule to find the exact value.

67. $\displaystyle\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$

68. $\displaystyle\lim_{x\to 0}\frac{5^x-4^x}{3^x-2^x}$

69–70 Illustrate l'Hospital's Rule by graphing both $f(x)/g(x)$ and $f'(x)/g'(x)$ near $x=0$ to see that these ratios have the same limit as $x\to 0$. Also, calculate the exact value of the limit.

69. $f(x)=e^x-1,\quad g(x)=x^3+4x$

70. $f(x)=2x\sin x,\quad g(x)=\sec x-1$

71. Prove that

$$\lim_{x\to\infty}\frac{e^x}{x^n}=\infty$$

for any positive integer n. This shows that the exponential function approaches infinity faster than any power of x.

72. Prove that

$$\lim_{x\to\infty}\frac{\ln x}{x^p}=0$$

for any number $p>0$. This shows that the logarithmic function approaches ∞ more slowly than any power of x.

73–74 What happens if you try to use l'Hospital's Rule to find the limit? Evaluate the limit using another method.

73. $\displaystyle\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}}$

74. $\displaystyle\lim_{x\to(\pi/2)^-}\frac{\sec x}{\tan x}$

75. Investigate the family of curves $f(x)=e^x-cx$. In particular, find the limits as $x\to\pm\infty$ and determine the values of c for which f has an absolute minimum. What happens to the minimum points as c increases?

76. If an object with mass m is dropped from rest, one model for its speed v after t seconds, taking air resistance into account, is

$$v=\frac{mg}{c}(1-e^{-ct/m})$$

where g is the acceleration due to gravity and c is a positive constant. (In Chapter 9 we will be able to deduce this equation from the assumption that the air resistance is proportional to the speed of the object; c is the proportionality constant.)

(a) Calculate $\lim_{t\to\infty}v$. What is the meaning of this limit?

(b) For fixed t, use l'Hospital's Rule to calculate $\lim_{c \to 0^+} v$. What can you conclude about the velocity of a falling object in a vacuum?

77. If an initial amount A_0 of money is invested at an interest rate r compounded n times a year, the value of the investment after t years is

$$A = A_0\left(1 + \frac{r}{n}\right)^{nt}$$

If we let $n \to \infty$, we refer to the *continuous compounding* of interest. Use l'Hospital's Rule to show that if interest is compounded continuously, then the amount after t years is

$$A = A_0 e^{rt}$$

78. If a metal ball with mass m is projected in water and the force of resistance is proportional to the square of the velocity, then the distance the ball travels in time t is

$$s(t) = \frac{m}{c} \ln \cosh \sqrt{\frac{gc}{mt}}$$

where c is a positive constant. Find $\lim_{c \to 0^+} s(t)$.

79. If an electrostatic field E acts on a liquid or a gaseous polar dielectric, the net dipole moment P per unit volume is

$$P(E) = \frac{e^E + e^{-E}}{e^E - e^{-E}} - \frac{1}{E}$$

Show that $\lim_{E \to 0^+} P(E) = 0$.

80. A metal cable has radius r and is covered by insulation, so that the distance from the center of the cable to the exterior of the insulation is R. The velocity v of an electrical impulse in the cable is

$$v = -c\left(\frac{r}{R}\right)^2 \ln\left(\frac{r}{R}\right)$$

where c is a positive constant. Find the following limits and interpret your answers.

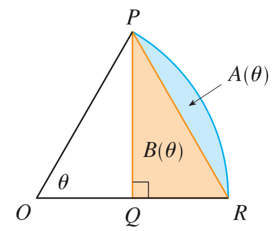
(a) $\lim_{R \to r^+} v$ (b) $\lim_{r \to 0^+} v$

81. The first appearance in print of l'Hospital's Rule was in the book *Analyse des Infiniment Petits* published by the Marquis de l'Hospital in 1696. This was the first calculus *textbook* ever published and the example that the Marquis used in that book to illustrate his rule was to find the limit of the function

$$y = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}}$$

as x approaches a, where $a > 0$. (At that time it was common to write aa instead of a^2.) Solve this problem.

82. The figure shows a sector of a circle with central angle θ. Let $A(\theta)$ be the area of the segment between the chord PR and the arc PR. Let $B(\theta)$ be the area of the triangle PQR. Find $\lim_{\theta \to 0^+} A(\theta)/B(\theta)$.



83. Evaluate $\displaystyle\lim_{x \to \infty} \left[x - x^2 \ln\left(\frac{1 + x}{x}\right)\right]$.

84. Suppose f is a positive function. If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$, show that

$$\lim_{x \to a} [f(x)]^{g(x)} = 0$$

This shows that 0^∞ is not an indeterminate form.

85. If f' is continuous, $f(2) = 0$, and $f'(2) = 7$, evaluate

$$\lim_{x \to 0} \frac{f(2 + 3x) + f(2 + 5x)}{x}$$

86. For what values of a and b is the following equation true?

$$\lim_{x \to 0} \left(\frac{\sin 2x}{x^3} + a + \frac{b}{x^2}\right) = 0$$

87. If f' is continuous, use l'Hospital's Rule to show that

$$\lim_{h \to 0} \frac{f(x + h) - f(x - h)}{2h} = f'(x)$$

Explain the meaning of this equation with the aid of a diagram.

88. If f'' is continuous, show that

$$\lim_{h \to 0} \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} = f''(x)$$

89. Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(a) Use the definition of derivative to compute $f'(0)$.
(b) Show that f has derivatives of all orders that are defined on $\mathbb{R}$. [*Hint:* First show by induction that there is a polynomial $p_n(x)$ and a nonnegative integer k_n such that $f^{(n)}(x) = p_n(x)f(x)/x^{k_n}$ for $x \neq 0$.]

90. Let

$$f(x) = \begin{cases} |x|^x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

(a) Show that f is continuous at 0.
(b) Investigate graphically whether f is differentiable at 0 by zooming in several times toward the point (0, 1) on the graph of f.
(c) Show that f is not differentiable at 0. How can you reconcile this fact with the appearance of the graphs in part (b)?