There is clearly no inflection point when $c \leqslant 1$. For $c > 1$ we calculate that

$$f''(x) = \frac{2(3x^2 + 6x + 4 - c)}{(x^2 + 2x + c)^3}$$

and deduce that inflection points occur when $x = -1 \pm \sqrt{3(c-1)}/3$. So the inflection points become more spread out as $c$ increases and this seems plausible from the last two parts of Figure 21.

## 4.6 Exercises

**1–8** Produce graphs of $f$ that reveal all the important aspects of the curve. In particular, you should use graphs of $f'$ and $f''$ to estimate the intervals of increase and decrease, extreme values, intervals of concavity, and inflection points.

**1.** $f(x) = 4x^4 - 32x^3 + 89x^2 - 95x + 29$

**2.** $f(x) = x^6 - 15x^5 + 75x^4 - 125x^3 - x$

**3.** $f(x) = x^6 - 10x^5 - 400x^4 + 2500x^3$

**4.** $f(x) = \dfrac{x^2 - 1}{40x^3 + x + 1}$ **5.** $f(x) = \dfrac{x}{x^3 + x^2 + 1}$

**6.** $f(x) = 6\sin x - x^2, \quad -5 \leqslant x \leqslant 3$

**7.** $f(x) = 6\sin x + \cot x, \quad -\pi \leqslant x \leqslant \pi$

**8.** $f(x) = e^x - 0.186x^4$

**9–10** Produce graphs of $f$ that reveal all the important aspects of the curve. Estimate the intervals of increase and decrease and intervals of concavity, and use calculus to find these intervals exactly.

**9.** $f(x) = 1 + \dfrac{1}{x} + \dfrac{8}{x^2} + \dfrac{1}{x^3}$ **10.** $f(x) = \dfrac{1}{x^8} - \dfrac{2 \times 10^8}{x^4}$

**11–12**
(a) Graph the function.
(b) Use l'Hospital's Rule to explain the behavior as $x \to 0$.
(c) Estimate the minimum value and intervals of concavity. Then use calculus to find the exact values.

**11.** $f(x) = x^2 \ln x$ **12.** $f(x) = xe^{1/x}$

**13–14** Sketch the graph by hand using asymptotes and intercepts, but not derivatives. Then use your sketch as a guide to producing graphs (with a graphing device) that display the major features of the curve. Use these graphs to estimate the maximum and minimum values.

**13.** $f(x) = \dfrac{(x+4)(x-3)^2}{x^4(x-1)}$ **14.** $f(x) = \dfrac{(2x+3)^2(x-2)^5}{x^3(x-5)^2}$

CAS **15.** If $f$ is the function considered in Example 3, use a computer algebra system to calculate $f'$ and then graph it to confirm that all the maximum and minimum values are as given in the example. Calculate $f''$ and use it to estimate the intervals of concavity and inflection points.

CAS **16.** If $f$ is the function of Exercise 14, find $f'$ and $f''$ and use their graphs to estimate the intervals of increase and decrease and concavity of $f$.

CAS **17–22** Use a computer algebra system to graph $f$ and to find $f'$ and $f''$. Use graphs of these derivatives to estimate the intervals of increase and decrease, extreme values, intervals of concavity, and inflection points of $f$.

**17.** $f(x) = \dfrac{x^3 + 5x^2 + 1}{x^4 + x^3 - x^2 + 2}$ **18.** $f(x) = \dfrac{x^{2/3}}{1 + x + x^4}$

**19.** $f(x) = \sqrt{x + 5\sin x}, \quad x \leqslant 20$

**20.** $f(x) = (x^2 - 1)e^{\arctan x}$

**21.** $f(x) = \dfrac{1 - e^{1/x}}{1 + e^{1/x}}$ **22.** $f(x) = \dfrac{1}{1 + e^{\tan x}}$

CAS **23–24** Graph the function using as many viewing rectangles as you need to depict the true nature of the function.

**23.** $f(x) = \dfrac{1 - \cos(x^4)}{x^8}$ **24.** $f(x) = e^x + \ln|x - 4|$

CAS **25–26**
(a) Graph the function.
(b) Explain the shape of the graph by computing the limit as $x \to 0^+$ or as $x \to \infty$.
(c) Estimate the maximum and minimum values and then use calculus to find the exact values.
(d) Use a graph of $f''$ to estimate the $x$-coordinates of the inflection points.

**25.** $f(x) = x^{1/x}$ **26.** $f(x) = (\sin x)^{\sin x}$

**27.** In Example 4 we considered a member of the family of functions $f(x) = \sin(x + \sin cx)$ that occur in FM synthesis. Here we investigate the function with $c = 3$. Start by graphing $f$ in the viewing rectangle $[0, \pi]$ by $[-1.2, 1.2]$. How many local maximum points do you see? The graph has more than are visible to the naked eye. To discover the hidden maximum and minimum points you will need to examine the graph of $f'$ very carefully. In fact, it helps to look at the graph of $f''$ at the same time. Find all the maximum and minimum values and inflection points. Then graph $f$ in the viewing rectangle $[-2\pi, 2\pi]$ by $[-1.2, 1.2]$ and comment on symmetry.

**28–35** Describe how the graph of $f$ varies as $c$ varies. Graph several members of the family to illustrate the trends that you discover. In particular, you should investigate how maximum and minimum points and inflection points move when $c$ changes. You should also identify any transitional values of $c$ at which the basic shape of the curve changes.

**28.** $f(x) = x^3 + cx$

**29.** $f(x) = \sqrt{x^4 + cx^2}$

**30.** $f(x) = x\sqrt{c^2 - x^2}$

**31.** $f(x) = e^x + ce^{-x}$

**32.** $f(x) = \ln(x^2 + c)$

**33.** $f(x) = \dfrac{cx}{1 + c^2x^2}$

**34.** $f(x) = x^2 + ce^{-x}$

**35.** $f(x) = cx + \sin x$

**36.** The family of functions $f(t) = C(e^{-at} - e^{-bt})$, where $a$, $b$, and $C$ are positive numbers and $b > a$, has been used to model the concentration of a drug injected into the bloodstream at time $t = 0$. Graph several members of this family. What do they have in common? For fixed values of $C$ and $a$, discover graphically what happens as $b$ increases. Then use calculus to prove what you have discovered.

**37.** Investigate the family of curves given by $f(x) = xe^{-cx}$, where $c$ is a real number. Start by computing the limits as $x \to \pm\infty$. Identify any transitional values of $c$ where the basic shape changes. What happens to the maximum or minimum points and inflection points as $c$ changes? Illustrate by graphing several members of the family.

**38.** Investigate the family of curves given by the equation $f(x) = x^4 + cx^2 + x$. Start by determining the transitional value of $c$ at which the number of inflection points changes. Then graph several members of the family to see what shapes are possible. There is another transitional value of $c$ at which the number of critical numbers changes. Try to discover it graphically. Then prove what you have discovered.

**39.** (a) Investigate the family of polynomials given by the equation $f(x) = cx^4 - 2x^2 + 1$. For what values of $c$ does the curve have minimum points?
(b) Show that the minimum and maximum points of every curve in the family lie on the parabola $y = 1 - x^2$. Illustrate by graphing this parabola and several members of the family.

**40.** (a) Investigate the family of polynomials given by the equation $f(x) = 2x^3 + cx^2 + 2x$. For what values of $c$ does the curve have maximum and minimum points?
(b) Show that the minimum and maximum points of every curve in the family lie on the curve $y = x - x^3$. Illustrate by graphing this curve and several members of the family.

## 4.7 Optimization Problems

The methods we have learned in this chapter for finding extreme values have practical applications in many areas of life. A businessperson wants to minimize costs and maximize profits. A traveler wants to minimize transportation time. Fermat's Principle in optics states that light follows the path that takes the least time. In this section we solve such problems as maximizing areas, volumes, and profits and minimizing distances, times, and costs.

In solving such practical problems the greatest challenge is often to convert the word problem into a mathematical optimization problem by setting up the function that is to be maximized or minimized. Let's recall the problem-solving principles discussed on page 75 and adapt them to this situation:

PS

### Steps in Solving Optimization Problems

1. **Understand the Problem** The first step is to read the problem carefully until it is clearly understood. Ask yourself: What is the unknown? What are the given quantities? What are the given conditions?
2. **Draw a Diagram** In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.