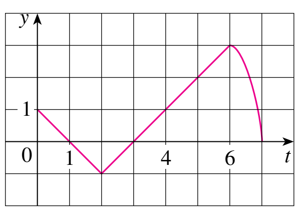
Before it was discovered, from the time of Eudoxus and Archimedes to the time of Galileo and Fermat, problems of finding areas, volumes, and lengths of curves were so difficult that only a genius could meet the challenge. But now, armed with the systematic method that Newton and Leibniz fashioned out of the Fundamental Theorem, we will see in the chapters to come that these challenging problems are accessible to all of us.

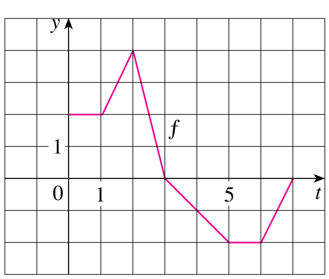
5.3 Exercises

1. Explain exactly what is meant by the statement that "differentiation and integration are inverse processes."

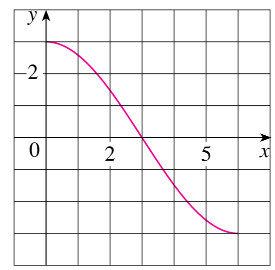
2. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.
(a) Evaluate $g(x)$ for $x = 0, 1, 2, 3, 4, 5$, and 6.
(b) Estimate $g(7)$.
(c) Where does g have a maximum value? Where does it have a minimum value?
(d) Sketch a rough graph of g.



3. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.
(a) Evaluate $g(0)$, $g(1)$, $g(2)$, $g(3)$, and $g(6)$.
(b) On what interval is g increasing?
(c) Where does g have a maximum value?
(d) Sketch a rough graph of g.



4. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.
(a) Evaluate $g(0)$ and $g(6)$.
(b) Estimate $g(x)$ for $x = 1, 2, 3, 4$, and 5.
(c) On what interval is g increasing?
(d) Where does g have a maximum value?
(e) Sketch a rough graph of g.
(f) Use the graph in part (e) to sketch the graph of $g'(x)$. Compare with the graph of f.



5–6 Sketch the area represented by $g(x)$. Then find $g'(x)$ in two ways: (a) by using Part 1 of the Fundamental Theorem and (b) by evaluating the integral using Part 2 and then differentiating.

5. $g(x) = \int_1^x t^2\,dt$

6. $g(x) = \int_0^x (2 + \sin t)\,dt$

7–18 Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of the function.

7. $g(x) = \int_1^x \dfrac{1}{t^3 + 1}\,dt$

8. $g(x) = \int_3^x e^{t^2 - t}\,dt$

9. $g(s) = \int_5^s (t - t^2)^8\,dt$

10. $g(r) = \int_0^r \sqrt{x^2 + 4}\,dx$

11. $F(x) = \int_x^\pi \sqrt{1 + \sec t}\,dt$

$\left[Hint\text{: } \int_x^\pi \sqrt{1 + \sec t}\,dt = -\int_\pi^x \sqrt{1 + \sec t}\,dt \right]$

12. $G(x) = \int_x^1 \cos\sqrt{t}\,dt$

13. $h(x) = \int_1^{e^x} \ln t\,dt$

14. $h(x) = \int_1^{\sqrt{x}} \dfrac{z^2}{z^4 + 1}\,dz$

15. $y = \int_0^{\tan x} \sqrt{t + \sqrt{t}}\,dt$

16. $y = \int_0^{x^4} \cos^2\theta\,d\theta$

17. $y = \int_{1-3x}^1 \dfrac{u^3}{1 + u^2}\,du$

18. $y = \int_{\sin x}^1 \sqrt{1 + t^2}\,dt$

Graphing calculator or computer required CAS Computer algebra system required 1. Homework Hints available at stewartcalculus.com

19–44 Evaluate the integral.

19. $\int_{-1}^{2} (x^3 - 2x)\,dx$

20. $\int_{-1}^{1} x^{100}\,dx$

21. $\int_{1}^{4} (5 - 2t + 3t^2)\,dt$

22. $\int_{0}^{1} \left(1 + \frac{1}{2}u^4 - \frac{2}{5}u^9\right) du$

23. $\int_{1}^{9} \sqrt{x}\,dx$

24. $\int_{1}^{8} x^{-2/3}\,dx$

25. $\int_{\pi/6}^{\pi} \sin\theta\,d\theta$

26. $\int_{-5}^{5} e\,dx$

27. $\int_{0}^{1} (u + 2)(u - 3)\,du$

28. $\int_{0}^{4} (4 - t)\sqrt{t}\,dt$

29. $\int_{1}^{9} \frac{x - 1}{\sqrt{x}}\,dx$

30. $\int_{0}^{2} (y - 1)(2y + 1)\,dy$

31. $\int_{0}^{\pi/4} \sec^2 t\,dt$

32. $\int_{0}^{\pi/4} \sec\theta\tan\theta\,d\theta$

33. $\int_{1}^{2} (1 + 2y)^2\,dy$

34. $\int_{0}^{3} (2\sin x - e^x)\,dx$

35. $\int_{1}^{2} \frac{v^3 + 3v^6}{v^4}\,dv$

36. $\int_{1}^{18} \sqrt{\frac{3}{z}}\,dz$

37. $\int_{0}^{1} (x^e + e^x)\,dx$

38. $\int_{0}^{1} \cosh t\,dt$

39. $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1 + x^2}\,dx$

40. $\int_{1}^{2} \frac{4 + u^2}{u^3}\,du$

41. $\int_{-1}^{1} e^{u+1}\,du$

42. $\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1 - x^2}}\,dx$

43. $\int_{0}^{\pi} f(x)\,dx$ where $f(x) = \begin{cases} \sin x & \text{if } 0 \leqslant x < \pi/2 \\ \cos x & \text{if } \pi/2 \leqslant x \leqslant \pi \end{cases}$

44. $\int_{-2}^{2} f(x)\,dx$ where $f(x) = \begin{cases} 2 & \text{if } -2 \leqslant x \leqslant 0 \\ 4 - x^2 & \text{if } 0 < x \leqslant 2 \end{cases}$

45–48 What is wrong with the equation?

45. $\int_{-2}^{1} x^{-4}\,dx = \left.\frac{x^{-3}}{-3}\right]_{-2}^{1} = -\frac{3}{8}$

46. $\int_{-1}^{2} \frac{4}{x^3}\,dx = -\left.\frac{2}{x^2}\right]_{-1}^{2} = \frac{3}{2}$

47. $\int_{\pi/3}^{\pi} \sec\theta\tan\theta\,d\theta = \sec\theta\Big]_{\pi/3}^{\pi} = -3$

48. $\int_{0}^{\pi} \sec^2 x\,dx = \tan x\Big]_{0}^{\pi} = 0$

49–52 Use a graph to give a rough estimate of the area of the region that lies beneath the given curve. Then find the exact area.

49. $y = \sqrt[3]{x},\quad 0 \leqslant x \leqslant 27$

50. $y = x^{-4},\quad 1 \leqslant x \leqslant 6$

51. $y = \sin x,\ 0 \leqslant x \leqslant \pi$

52. $y = \sec^2 x,\ 0 \leqslant x \leqslant \pi/3$

53–54 Evaluate the integral and interpret it as a difference of areas. Illustrate with a sketch.

53. $\int_{-1}^{2} x^3\,dx$

54. $\int_{\pi/6}^{2\pi} \cos x\,dx$

55–59 Find the derivative of the function.

55. $g(x) = \int_{2x}^{3x} \frac{u^2 - 1}{u^2 + 1}\,du$

$\left[\textit{Hint: } \int_{2x}^{3x} f(u)\,du = \int_{2x}^{0} f(u)\,du + \int_{0}^{3x} f(u)\,du\right]$

56. $g(x) = \int_{1-2x}^{1+2x} t\sin t\,dt$

57. $F(x) = \int_{x}^{x^2} e^{t^2}\,dt$

58. $F(x) = \int_{\sqrt{x}}^{2x} \arctan t\,dt$

59. $y = \int_{\cos x}^{\sin x} \ln(1 + 2v)\,dv$

60. If $f(x) = \int_0^x (1 - t^2)e^{t^2}\,dt$, on what interval is f increasing?

61. On what interval is the curve

$$y = \int_0^x \frac{t^2}{t^2 + t + 2}\,dt$$

concave downward?

62. If $f(x) = \int_0^{\sin x} \sqrt{1 + t^2}\,dt$ and $g(y) = \int_3^y f(x)\,dx$, find $g''(\pi/6)$.

63. If $f(1) = 12$, f' is continuous, and $\int_1^4 f'(x)\,dx = 17$, what is the value of $f(4)$?

64. The **error function**

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

is used in probability, statistics, and engineering.

(a) Show that $\int_a^b e^{-t^2}\,dt = \frac{1}{2}\sqrt{\pi}\,[\operatorname{erf}(b) - \operatorname{erf}(a)]$.

(b) Show that the function $y = e^{x^2}\operatorname{erf}(x)$ satisfies the differential equation $y' = 2xy + 2/\sqrt{\pi}$.

65. The Fresnel function S was defined in Example 3 and graphed in Figures 7 and 8.

(a) At what values of x does this function have local maximum values?

(b) On what intervals is the function concave upward?

CAS (c) Use a graph to solve the following equation correct to two decimal places:

$$\int_0^x \sin(\pi t^2/2)\,dt = 0.2$$

CAS **66.** The **sine integral function**

$$\text{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt$$

is important in electrical engineering. [The integrand $f(t) = (\sin t)/t$ is not defined when $t = 0$, but we know that its limit is 1 when $t \to 0$. So we define $f(0) = 1$ and this makes f a continuous function everywhere.]

(a) Draw the graph of Si.

(b) At what values of x does this function have local maximum values?

(c) Find the coordinates of the first inflection point to the right of the origin.

(d) Does this function have horizontal asymptotes?

(e) Solve the following equation correct to one decimal place:

$$\int_0^x \frac{\sin t}{t}\,dt = 1$$

67–68 Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.

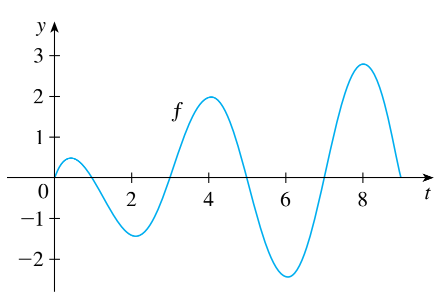
(a) At what values of x do the local maximum and minimum values of g occur?

(b) Where does g attain its absolute maximum value?

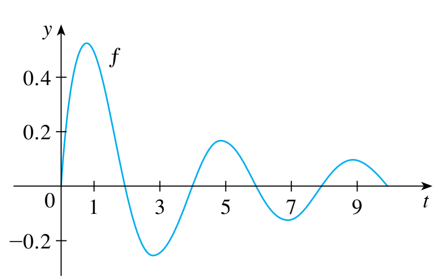
(c) On what intervals is g concave downward?

(d) Sketch the graph of g.

67.



68.



69–70 Evaluate the limit by first recognizing the sum as a Riemann sum for a function defined on [0, 1].

69. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^4}$

70. $\displaystyle \lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \sqrt{\frac{3}{n}} + \cdots + \sqrt{\frac{n}{n}}\right)$

71. Justify $\boxed{3}$ for the case $h < 0$.

72. If f is continuous and g and h are differentiable functions, find a formula for

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt$$

73. (a) Show that $1 \leqslant \sqrt{1 + x^3} \leqslant 1 + x^3$ for $x \geqslant 0$.

(b) Show that $1 \leqslant \int_0^1 \sqrt{1 + x^3}\,dx \leqslant 1.25$.

74. (a) Show that $\cos(x^2) \geqslant \cos x$ for $0 \leqslant x \leqslant 1$.

(b) Deduce that $\int_0^{\pi/6} \cos(x^2)\,dx \geqslant \frac{1}{2}$.

75. Show that

$$0 \leqslant \int_5^{10} \frac{x^2}{x^4 + x^2 + 1}\,dx \leqslant 0.1$$

by comparing the integrand to a simpler function.

76. Let

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leqslant x \leqslant 1 \\ 2 - x & \text{if } 1 < x \leqslant 2 \\ 0 & \text{if } x > 2 \end{cases}$$

and $\qquad g(x) = \int_0^x f(t)\,dt$

(a) Find an expression for $g(x)$ similar to the one for $f(x)$.

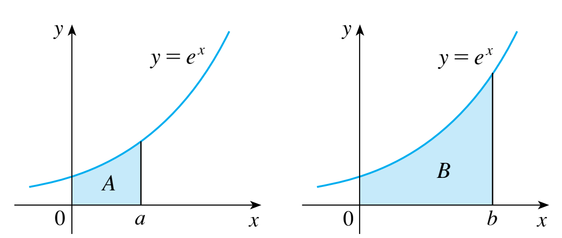
(b) Sketch the graphs of f and g.

(c) Where is f differentiable? Where is g differentiable?

77. Find a function f and a number a such that

$$6 + \int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x} \qquad \text{for all } x > 0$$

78. The area labeled B is three times the area labeled A. Express b in terms of a.



79. A manufacturing company owns a major piece of equipment that depreciates at the (continuous) rate $f = f(t)$, where t is the time measured in months since its last overhaul. Because a fixed cost A is incurred each time the machine is overhauled, the company wants to determine the optimal time T (in months) between overhauls.

(a) Explain why $\int_0^t f(s)\,ds$ represents the loss in value of the machine over the period of time t since the last overhaul.

(b) Let $C = C(t)$ be given by

$$C(t) = \frac{1}{t}\left[A + \int_0^t f(s)\,ds\right]$$

What does C represent and why would the company want to minimize C?

(c) Show that C has a minimum value at the numbers $t = T$ where $C(T) = f(T)$.

80. A high-tech company purchases a new computing system whose initial value is V. The system will depreciate at the rate $f = f(t)$ and will accumulate maintenance costs at the rate $g = g(t)$, where t is the time measured in months. The company wants to determine the optimal time to replace the system.

(a) Let

$$C(t) = \frac{1}{t}\int_0^t [f(s) + g(s)]\,ds$$

Show that the critical numbers of C occur at the numbers t where $C(t) = f(t) + g(t)$.

(b) Suppose that

$$f(t) = \begin{cases} \dfrac{V}{15} - \dfrac{V}{450}t & \text{if } 0 < t \leqslant 30 \\ 0 & \text{if } t > 30 \end{cases}$$

and $$g(t) = \frac{Vt^2}{12{,}900} \qquad t > 0$$

Determine the length of time T for the total depreciation $D(t) = \int_0^t f(s)\,ds$ to equal the initial value V.

(c) Determine the absolute minimum of C on $(0, T]$.

(d) Sketch the graphs of C and $f + g$ in the same coordinate system, and verify the result in part (a) in this case.

5.4 Indefinite Integrals and the Net Change Theorem

We saw in Section 5.3 that the second part of the Fundamental Theorem of Calculus provides a very powerful method for evaluating the definite integral of a function, assuming that we can find an antiderivative of the function. In this section we introduce a notation for antiderivatives, review the formulas for antiderivatives, and use them to evaluate definite integrals. We also reformulate FTC2 in a way that makes it easier to apply to science and engineering problems.

Indefinite Integrals

Both parts of the Fundamental Theorem establish connections between antiderivatives and definite integrals. Part 1 says that if f is continuous, then $\int_a^x f(t)\,dt$ is an antiderivative of f. Part 2 says that $\int_a^b f(x)\,dx$ can be found by evaluating $F(b) - F(a)$, where F is an antiderivative of f.

We need a convenient notation for antiderivatives that makes them easy to work with. Because of the relation given by the Fundamental Theorem between antiderivatives and integrals, the notation $\int f(x)\,dx$ is traditionally used for an antiderivative of f and is called an **indefinite integral**. Thus

$$\int f(x)\,dx = F(x) \qquad \text{means} \qquad F'(x) = f(x)$$

For example, we can write

$$\int x^2\,dx = \frac{x^3}{3} + C \qquad \text{because} \qquad \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$$