

so integration by parts gives

$$\int \sin^n x \, dx = -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx$$

Since  $\cos^2 x = 1 - \sin^2 x$ , we have

$$\int \sin^n x \, dx = -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$$

As in Example 4, we solve this equation for the desired integral by taking the last term on the right side to the left side. Thus we have

$$n \int \sin^n x \, dx = -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, dx$$

$$\text{or} \quad \int \sin^n x \, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

The reduction formula [7] is useful because by using it repeatedly we could eventually express  $\int \sin^n x \, dx$  in terms of  $\int \sin x \, dx$  (if  $n$  is odd) or  $\int (\sin x)^0 \, dx = \int dx$  (if  $n$  is even).

## 7.1 Exercises

**1–2** Evaluate the integral using integration by parts with the indicated choices of  $u$  and  $dv$ .

1.  $\int x^2 \ln x \, dx$ ;  $u = \ln x$ ,  $dv = x^2 \, dx$

2.  $\int \theta \cos \theta \, d\theta$ ;  $u = \theta$ ,  $dv = \cos \theta \, d\theta$

**3–36** Evaluate the integral.

3.  $\int x \cos 5x \, dx$

4.  $\int ye^{0.2y} \, dy$

5.  $\int te^{-3t} \, dt$

6.  $\int (x-1) \sin \pi x \, dx$

7.  $\int (x^2 + 2x) \cos x \, dx$

8.  $\int t^2 \sin \beta t \, dt$

9.  $\int \ln \sqrt[3]{x} \, dx$

10.  $\int \sin^{-1} x \, dx$

11.  $\int \arctan 4t \, dt$

12.  $\int p^5 \ln p \, dp$

13.  $\int t \sec^2 2t \, dt$

15.  $\int (\ln x)^2 \, dx$

17.  $\int e^{2\theta} \sin 3\theta \, d\theta$

19.  $\int z^3 e^z \, dz$

21.  $\int \frac{xe^{2x}}{(1+2x)^2} \, dx$

23.  $\int_0^{1/2} x \cos \pi x \, dx$

25.  $\int_0^1 t \cosh t \, dt$

27.  $\int_1^3 r^3 \ln r \, dr$

14.  $\int s 2^s \, ds$

16.  $\int t \sinh mt \, dt$

18.  $\int e^{-\theta} \cos 2\theta \, d\theta$

20.  $\int x \tan^2 x \, dx$

22.  $\int (\arcsin x)^2 \, dx$

24.  $\int_0^1 (x^2 + 1)e^{-x} \, dx$


26.  $\int_4^9 \frac{\ln y}{\sqrt{y}} \, dy$

28.  $\int_0^{2\pi} t^2 \sin 2t \, dt$

29.  $\int_0^1 \frac{y}{e^{2y}} dy$       30.  $\int_1^{\sqrt{3}} \arctan(1/x) dx$
31.  $\int_0^{1/2} \cos^{-1}x dx$       32.  $\int_1^2 \frac{(\ln x)^2}{x^3} dx$
33.  $\int \cos x \ln(\sin x) dx$       34.  $\int_0^1 \frac{r^3}{\sqrt{4+r^2}} dr$
35.  $\int_1^2 x^4(\ln x)^2 dx$       36.  $\int_0^t e^s \sin(t-s) ds$

**37–42** First make a substitution and then use integration by parts to evaluate the integral.

37.  $\int \cos \sqrt{x} dx$       38.  $\int t^3 e^{-t^2} dt$
39.  $\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2) d\theta$       40.  $\int_0^{\pi} e^{\cos t} \sin 2t dt$
41.  $\int x \ln(1+x) dx$       42.  $\int \sin(\ln x) dx$

 **43–46** Evaluate the indefinite integral. Illustrate, and check that your answer is reasonable, by graphing both the function and its antiderivative (take  $C = 0$ ).

43.  $\int x e^{-2x} dx$       44.  $\int x^{3/2} \ln x dx$
45.  $\int x^3 \sqrt{1+x^2} dx$       46.  $\int x^2 \sin 2x dx$

**47.** (a) Use the reduction formula in Example 6 to show that

$$\int \sin^2 x dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) Use part (a) and the reduction formula to evaluate  $\int \sin^4 x dx$ .

**48.** (a) Prove the reduction formula

$$\int \cos^n x dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx$$

(b) Use part (a) to evaluate  $\int \cos^2 x dx$ .

(c) Use parts (a) and (b) to evaluate  $\int \cos^4 x dx$ .

**49.** (a) Use the reduction formula in Example 6 to show that

$$\int_0^{\pi/2} \sin^n x dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x dx$$

where  $n \geq 2$  is an integer.

(b) Use part (a) to evaluate  $\int_0^{\pi/2} \sin^3 x dx$  and  $\int_0^{\pi/2} \sin^5 x dx$ .

(c) Use part (a) to show that, for odd powers of sine,

$$\int_0^{\pi/2} \sin^{2n+1} x dx = \frac{2 \cdot 4 \cdot 6 \cdots \cdot 2n}{3 \cdot 5 \cdot 7 \cdots \cdot (2n+1)}$$

**50.** Prove that, for even powers of sine,

$$\int_0^{\pi/2} \sin^{2n} x dx = \frac{1 \cdot 3 \cdot 5 \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdots \cdot 2n} \frac{\pi}{2}$$

**51–54** Use integration by parts to prove the reduction formula.

**51.**  $\int (\ln x)^n dx = x(\ln x)^n - n \int (\ln x)^{n-1} dx$

**52.**  $\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$

**53.**  $\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx \quad (n \neq 1)$


**54.**  $\int \sec^n x dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx \quad (n \neq 1)$

**55.** Use Exercise 51 to find  $\int (\ln x)^3 dx$ .

**56.** Use Exercise 52 to find  $\int x^4 e^x dx$ .

**57–58** Find the area of the region bounded by the given curves.

**57.**  $y = x^2 \ln x$ ,  $y = 4 \ln x$       **58.**  $y = x^2 e^{-x}$ ,  $y = x e^{-x}$

 **59–60** Use a graph to find approximate  $x$ -coordinates of the points of intersection of the given curves. Then find (approximately) the area of the region bounded by the curves.

**59.**  $y = \arcsin(\frac{1}{2}x)$ ,  $y = 2 - x^2$

**60.**  $y = x \ln(x+1)$ ,  $y = 3x - x^2$

**61–63** Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the specified axis.

**61.**  $y = \cos(\pi x/2)$ ,  $y = 0$ ,  $0 \leq x \leq 1$ ; about the  $y$ -axis

**62.**  $y = e^x$ ,  $y = e^{-x}$ ,  $x = 1$ ; about the  $y$ -axis

**63.**  $y = e^{-x}$ ,  $y = 0$ ,  $x = -1$ ,  $x = 0$ ; about  $x = 1$

**64.** Calculate the volume generated by rotating the region bounded by the curves  $y = \ln x$ ,  $y = 0$ , and  $x = 2$  about each axis.

(a) the  $y$ -axis

(b) the  $x$ -axis

65. Calculate the average value of  $f(x) = x \sec^2 x$  on the interval  $[0, \pi/4]$ .
66. A rocket accelerates by burning its onboard fuel, so its mass decreases with time. Suppose the initial mass of the rocket at liftoff (including its fuel) is  $m$ , the fuel is consumed at rate  $r$ , and the exhaust gases are ejected with constant velocity  $v_e$  (relative to the rocket). A model for the velocity of the rocket at time  $t$  is given by the equation

$$v(t) = -gt - v_e \ln \frac{m - rt}{m}$$

where  $g$  is the acceleration due to gravity and  $t$  is not too large. If  $g = 9.8 \text{ m/s}^2$ ,  $m = 30,000 \text{ kg}$ ,  $r = 160 \text{ kg/s}$ , and  $v_e = 3000 \text{ m/s}$ , find the height of the rocket one minute after liftoff.

67. A particle that moves along a straight line has velocity  $v(t) = t^2 e^{-t}$  meters per second after  $t$  seconds. How far will it travel during the first  $t$  seconds?
68. If  $f(0) = g(0) = 0$  and  $f''$  and  $g''$  are continuous, show that

$$\int_0^a f(x)g''(x) dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x) dx$$

69. Suppose that  $f(1) = 2$ ,  $f(4) = 7$ ,  $f'(1) = 5$ ,  $f'(4) = 3$ , and  $f''$  is continuous. Find the value of  $\int_1^4 x f''(x) dx$ .

70. (a) Use integration by parts to show that

$$\int f(x) dx = xf(x) - \int xf'(x) dx$$

- (b) If  $f$  and  $g$  are inverse functions and  $f'$  is continuous, prove that

$$\int_a^b f(x) dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y) dy$$

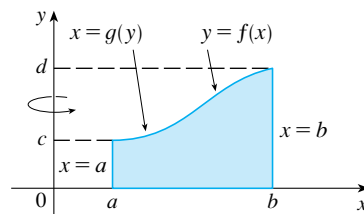
[Hint: Use part (a) and make the substitution  $y = f(x)$ .]

- (c) In the case where  $f$  and  $g$  are positive functions and  $b > a > 0$ , draw a diagram to give a geometric interpretation of part (b).
- (d) Use part (b) to evaluate  $\int_1^e \ln x dx$ .
71. We arrived at Formula 6.3.2,  $V = \int_a^b 2\pi x f(x) dx$ , by using cylindrical shells, but now we can use integration by parts to prove it using the slicing method of Section 6.2, at least for the case where  $f$  is one-to-one and therefore has an inverse function  $g$ . Use the figure to show that

$$V = \pi b^2 d - \pi a^2 c - \int_c^d \pi [g(y)]^2 dy$$

Make the substitution  $y = f(x)$  and then use integration by parts on the resulting integral to prove that

$$V = \int_a^b 2\pi x f(x) dx$$



72. Let  $I_n = \int_0^{\pi/2} \sin^n x dx$ .

- (a) Show that  $I_{2n+2} \leq I_{2n+1} \leq I_{2n}$ .
- (b) Use Exercise 50 to show that

$$\frac{I_{2n+2}}{I_{2n}} = \frac{2n+1}{2n+2}$$

- (c) Use parts (a) and (b) to show that

$$\frac{2n+1}{2n+2} \leq \frac{I_{2n+1}}{I_{2n}} \leq 1$$

and deduce that  $\lim_{n \rightarrow \infty} I_{2n+1}/I_{2n} = 1$ .

- (d) Use part (c) and Exercises 49 and 50 to show that

$$\lim_{n \rightarrow \infty} \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \dots \cdot \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} = \frac{\pi}{2}$$

This formula is usually written as an infinite product:

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \dots$$

and is called the *Wallis product*.

- (e) We construct rectangles as follows. Start with a square of area 1 and attach rectangles of area 1 alternately beside or on top of the previous rectangle (see the figure). Find the limit of the ratios of width to height of these rectangles.

