Then

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx$$

$$= \sec x \tan x - \int \sec x\,(\sec^2 x - 1)\,dx$$

$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

Using Formula 1 and solving for the required integral, we get

$$\int \sec^3 x\,dx = \tfrac{1}{2}\big(\sec x \tan x + \ln|\sec x + \tan x|\big) + C$$

Integrals such as the one in the preceding example may seem very special but they occur frequently in applications of integration, as we will see in Chapter 8. Integrals of the form $\int \cot^m x \csc^n x\,dx$ can be found by similar methods because of the identity $1 + \cot^2 x = \csc^2 x$.

Finally, we can make use of another set of trigonometric identities:

These product identities are discussed in Appendix D.

2 To evaluate the integrals (a) $\int \sin mx \cos nx\,dx$, (b) $\int \sin mx \sin nx\,dx$, or (c) $\int \cos mx \cos nx\,dx$, use the corresponding identity:

(a) $\sin A \cos B = \tfrac{1}{2}[\sin(A - B) + \sin(A + B)]$

(b) $\sin A \sin B = \tfrac{1}{2}[\cos(A - B) - \cos(A + B)]$

(c) $\cos A \cos B = \tfrac{1}{2}[\cos(A - B) + \cos(A + B)]$

EXAMPLE 9 Evaluate $\int \sin 4x \cos 5x\,dx$.

SOLUTION This integral could be evaluated using integration by parts, but it's easier to use the identity in Equation 2(a) as follows:

$$\int \sin 4x \cos 5x\,dx = \int \tfrac{1}{2}[\sin(-x) + \sin 9x]\,dx$$

$$= \tfrac{1}{2}\int (-\sin x + \sin 9x)\,dx$$

$$= \tfrac{1}{2}\big(\cos x - \tfrac{1}{9}\cos 9x\big) + C$$

7.2 Exercises

1–49 Evaluate the integral.

1. $\int \sin^2 x \cos^3 x\,dx$

2. $\int \sin^3\theta \cos^4\theta\,d\theta$

3. $\int_0^{\pi/2} \sin^7\theta \cos^5\theta\,d\theta$

4. $\int_0^{\pi/2} \sin^5 x\,dx$

5. $\int \sin^2(\pi x) \cos^5(\pi x)\,dx$

6. $\int \dfrac{\sin^3(\sqrt{x})}{\sqrt{x}}\,dx$

7. $\int_0^{\pi/2} \cos^2\theta\,d\theta$

8. $\int_0^{2\pi} \sin^2\big(\tfrac{1}{3}\theta\big)\,d\theta$

9. $\int_0^{\pi} \cos^4(2t)\,dt$

10. $\int_0^{\pi} \sin^2 t \cos^4 t\,dt$

11. $\int_0^{\pi/2} \sin^2 x \cos^2 x\,dx$

12. $\int_0^{\pi/2} (2 - \sin\theta)^2\,d\theta$

13. $\int t \sin^2 t\,dt$

14. $\int \cos\theta \cos^5(\sin\theta)\,d\theta$

15. $\int \dfrac{\cos^5\alpha}{\sqrt{\sin\alpha}}\,d\alpha$

16. $\int x \sin^3 x\,dx$

Graphing calculator or computer required **1.** Homework Hints available at stewartcalculus.com

17. $\int \cos^2 x \tan^3 x \, dx$

18. $\int \cot^5 \theta \sin^4 \theta \, d\theta$

19. $\int \dfrac{\cos x + \sin 2x}{\sin x} \, dx$

20. $\int \cos^2 x \sin 2x \, dx$

21. $\int \tan x \sec^3 x \, dx$

22. $\int \tan^2 \theta \sec^4 \theta \, d\theta$

23. $\int \tan^2 x \, dx$

24. $\int (\tan^2 x + \tan^4 x) \, dx$

25. $\int \tan^4 x \sec^6 x \, dx$

26. $\int_0^{\pi/4} \sec^4 \theta \tan^4 \theta \, d\theta$

27. $\int_0^{\pi/3} \tan^5 x \sec^4 x \, dx$

28. $\int \tan^5 x \sec^3 x \, dx$

29. $\int \tan^3 x \sec x \, dx$

30. $\int_0^{\pi/4} \tan^4 t \, dt$

31. $\int \tan^5 x \, dx$

32. $\int \tan^2 x \sec x \, dx$

33. $\int x \sec x \tan x \, dx$

34. $\int \dfrac{\sin \phi}{\cos^3 \phi} \, d\phi$

35. $\int_{\pi/6}^{\pi/2} \cot^2 x \, dx$

36. $\int_{\pi/4}^{\pi/2} \cot^3 x \, dx$

37. $\int_{\pi/4}^{\pi/2} \cot^5 \phi \csc^3 \phi \, d\phi$

38. $\int \csc^4 x \cot^6 x \, dx$

39. $\int \csc x \, dx$

40. $\int_{\pi/6}^{\pi/3} \csc^3 x \, dx$

41. $\int \sin 8x \cos 5x \, dx$

42. $\int \cos \pi x \cos 4\pi x \, dx$

43. $\int \sin 5\theta \sin \theta \, d\theta$

44. $\int \dfrac{\cos x + \sin x}{\sin 2x} \, dx$

45. $\int_0^{\pi/6} \sqrt{1 + \cos 2x} \, dx$

46. $\int_0^{\pi/4} \sqrt{1 - \cos 4\theta} \, d\theta$

47. $\int \dfrac{1 - \tan^2 x}{\sec^2 x} \, dx$

48. $\int \dfrac{dx}{\cos x - 1}$

49. $\int x \tan^2 x \, dx$

50. If $\int_0^{\pi/4} \tan^6 x \sec x \, dx = I$, express the value of $\int_0^{\pi/4} \tan^8 x \sec x \, dx$ in terms of I.

51–54 Evaluate the indefinite integral. Illustrate, and check that your answer is reasonable, by graphing both the integrand and its antiderivative (taking $C = 0$).

51. $\int x \sin^2(x^2) \, dx$

52. $\int \sin^5 x \cos^3 x \, dx$

53. $\int \sin 3x \sin 6x \, dx$

54. $\int \sec^4 \dfrac{x}{2} \, dx$

55. Find the average value of the function $f(x) = \sin^2 x \cos^3 x$ on the interval $[-\pi, \pi]$.

56. Evaluate $\int \sin x \cos x \, dx$ by four methods:
(a) the substitution $u = \cos x$
(b) the substitution $u = \sin x$
(c) the identity $\sin 2x = 2 \sin x \cos x$
(d) integration by parts
Explain the different appearances of the answers.

57–58 Find the area of the region bounded by the given curves.

57. $y = \sin^2 x, \quad y = \cos^2 x, \quad -\pi/4 \leqslant x \leqslant \pi/4$

58. $y = \sin^3 x, \quad y = \cos^3 x, \quad \pi/4 \leqslant x \leqslant 5\pi/4$

59–60 Use a graph of the integrand to guess the value of the integral. Then use the methods of this section to prove that your guess is correct.

59. $\int_0^{2\pi} \cos^3 x \, dx$

60. $\int_0^2 \sin 2\pi x \cos 5\pi x \, dx$

61–64 Find the volume obtained by rotating the region bounded by the given curves about the specified axis.

61. $y = \sin x, \ y = 0, \ \pi/2 \leqslant x \leqslant \pi;$ about the x-axis

62. $y = \sin^2 x, \ y = 0, \ 0 \leqslant x \leqslant \pi;$ about the x-axis

63. $y = \sin x, \ y = \cos x, \ 0 \leqslant x \leqslant \pi/4;$ about $y = 1$

64. $y = \sec x, \ y = \cos x, \ 0 \leqslant x \leqslant \pi/3;$ about $y = -1$

65. A particle moves on a straight line with velocity function $v(t) = \sin \omega t \cos^2 \omega t$. Find its position function $s = f(t)$ if $f(0) = 0$.

66. Household electricity is supplied in the form of alternating current that varies from 155 V to -155 V with a frequency of 60 cycles per second (Hz). The voltage is thus given by the equation

$$E(t) = 155 \sin(120\pi t)$$

where t is the time in seconds. Voltmeters read the RMS (root-mean-square) voltage, which is the square root of the average value of $[E(t)]^2$ over one cycle.
(a) Calculate the RMS voltage of household current.
(b) Many electric stoves require an RMS voltage of 220 V. Find the corresponding amplitude A needed for the voltage $E(t) = A \sin(120\pi t)$.

67–69 Prove the formula, where m and n are positive integers.

67. $\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$

68. $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$

69. $\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$

70. A *finite Fourier series* is given by the sum

$$\begin{aligned} f(x) &= \sum_{n=1}^{N} a_n \sin nx \\ &= a_1 \sin x + a_2 \sin 2x + \cdots + a_N \sin Nx \end{aligned}$$

Show that the mth coefficient a_m is given by the formula

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx$$

7.3 Trigonometric Substitution

In finding the area of a circle or an ellipse, an integral of the form $\int \sqrt{a^2 - x^2}\, dx$ arises, where $a > 0$. If it were $\int x\sqrt{a^2 - x^2}\, dx$, the substitution $u = a^2 - x^2$ would be effective but, as it stands, $\int \sqrt{a^2 - x^2}\, dx$ is more difficult. If we change the variable from x to θ by the substitution $x = a \sin \theta$, then the identity $1 - \sin^2\theta = \cos^2\theta$ allows us to get rid of the root sign because

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2 \cos^2\theta} = a|\cos \theta|$$

Notice the difference between the substitution $u = a^2 - x^2$ (in which the new variable is a function of the old one) and the substitution $x = a \sin \theta$ (the old variable is a function of the new one).

In general, we can make a substitution of the form $x = g(t)$ by using the Substitution Rule in reverse. To make our calculations simpler, we assume that g has an inverse function; that is, g is one-to-one. In this case, if we replace u by x and x by t in the Substitution Rule (Equation 5.5.4), we obtain

$$\int f(x)\, dx = \int f(g(t))g'(t)\, dt$$

This kind of substitution is called *inverse substitution.*

We can make the inverse substitution $x = a \sin \theta$ provided that it defines a one-to-one function. This can be accomplished by restricting θ to lie in the interval $[-\pi/2, \pi/2]$.

In the following table we list trigonometric substitutions that are effective for the given radical expressions because of the specified trigonometric identities. In each case the restriction on θ is imposed to ensure that the function that defines the substitution is one-to-one. (These are the same intervals used in Section 1.6 in defining the inverse functions.)

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$