Figure 5 shows the graphs of the integrand in Example 7 and its indefinite integral (with $C = 0$). Which is which?

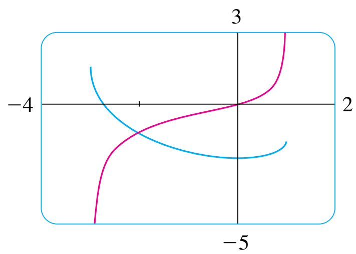


FIGURE 5

We now substitute $u = 2\sin\theta$, giving $du = 2\cos\theta\,d\theta$ and $\sqrt{4-u^2} = 2\cos\theta$, so

$$\begin{aligned}\int \frac{x}{\sqrt{3-2x-x^2}}\,dx &= \int \frac{2\sin\theta - 1}{2\cos\theta}\,2\cos\theta\,d\theta \\ &= \int (2\sin\theta - 1)\,d\theta \\ &= -2\cos\theta - \theta + C \\ &= -\sqrt{4-u^2} - \sin^{-1}\!\left(\frac{u}{2}\right) + C \\ &= -\sqrt{3-2x-x^2} - \sin^{-1}\!\left(\frac{x+1}{2}\right) + C\end{aligned}$$

7.3 Exercises

1–3 Evaluate the integral using the indicated trigonometric substitution. Sketch and label the associated right triangle.

1. $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$ $x = 2\sin\theta$

2. $\displaystyle\int \frac{x^3}{\sqrt{x^2+4}}\,dx$ $x = 2\tan\theta$

3. $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\,dx$ $x = 2\sec\theta$

4–30 Evaluate the integral.

4. $\displaystyle\int_0^1 x^3\sqrt{1-x^2}\,dx$

5. $\displaystyle\int_{\sqrt{2}}^2 \frac{1}{t^3\sqrt{t^2-1}}\,dt$

6. $\displaystyle\int_0^3 \frac{x}{\sqrt{36-x^2}}\,dx$

7. $\displaystyle\int_0^a \frac{dx}{(a^2+x^2)^{3/2}}$, $a > 0$

8. $\displaystyle\int \frac{dt}{t^2\sqrt{t^2-16}}$

9. $\displaystyle\int \frac{dx}{\sqrt{x^2+16}}$

10. $\displaystyle\int \frac{t^5}{\sqrt{t^2+2}}\,dt$

11. $\displaystyle\int \sqrt{1-4x^2}\,dx$

12. $\displaystyle\int \frac{du}{u\sqrt{5-u^2}}$

13. $\displaystyle\int \frac{\sqrt{x^2-9}}{x^3}\,dx$

14. $\displaystyle\int_0^1 \frac{dx}{(x^2+1)^2}$

15. $\displaystyle\int_0^a x^2\sqrt{a^2-x^2}\,dx$

16. $\displaystyle\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{9x^2-1}}$

17. $\displaystyle\int \frac{x}{\sqrt{x^2-7}}\,dx$

18. $\displaystyle\int \frac{dx}{[(ax)^2-b^2]^{3/2}}$

19. $\displaystyle\int \frac{\sqrt{1+x^2}}{x}\,dx$

20. $\displaystyle\int \frac{x}{\sqrt{1+x^2}}\,dx$

21. $\displaystyle\int_0^{0.6} \frac{x^2}{\sqrt{9-25x^2}}\,dx$

22. $\displaystyle\int_0^1 \sqrt{x^2+1}\,dx$

23. $\displaystyle\int \sqrt{5+4x-x^2}\,dx$

24. $\displaystyle\int \frac{dt}{\sqrt{t^2-6t+13}}$

25. $\displaystyle\int \frac{x}{\sqrt{x^2+x+1}}\,dx$

26. $\displaystyle\int \frac{x^2}{(3+4x-4x^2)^{3/2}}\,dx$

27. $\displaystyle\int \sqrt{x^2+2x}\,dx$

28. $\displaystyle\int \frac{x^2+1}{(x^2-2x+2)^2}\,dx$

29. $\displaystyle\int x\sqrt{1-x^4}\,dx$

30. $\displaystyle\int_0^{\pi/2} \frac{\cos t}{\sqrt{1+\sin^2 t}}\,dt$

31. (a) Use trigonometric substitution to show that

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x + \sqrt{x^2+a^2}\right) + C$$

(b) Use the hyperbolic substitution $x = a\sinh t$ to show that

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\!\left(\frac{x}{a}\right) + C$$

These formulas are connected by Formula 3.11.3.

32. Evaluate

$$\int \frac{x^2}{(x^2+a^2)^{3/2}}\,dx$$

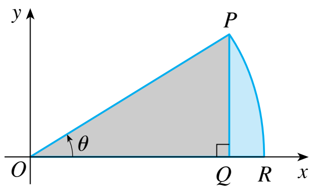
(a) by trigonometric substitution.
(b) by the hyperbolic substitution $x = a\sinh t$.

33. Find the average value of $f(x) = \sqrt{x^2-1}/x$, $1 \leqslant x \leqslant 7$.

34. Find the area of the region bounded by the hyperbola $9x^2 - 4y^2 = 36$ and the line $x = 3$.

Graphing calculator or computer required   1. Homework Hints available at stewartcalculus.com

35. Prove the formula $A = \frac{1}{2}r^2\theta$ for the area of a sector of a circle with radius r and central angle θ. [*Hint:* Assume $0 < \theta < \pi/2$ and place the center of the circle at the origin so it has the equation $x^2 + y^2 = r^2$. Then A is the sum of the area of the triangle POQ and the area of the region PQR in the figure.]



36. Evaluate the integral

$$\int \frac{dx}{x^4\sqrt{x^2 - 2}}$$

Graph the integrand and its indefinite integral on the same screen and check that your answer is reasonable.

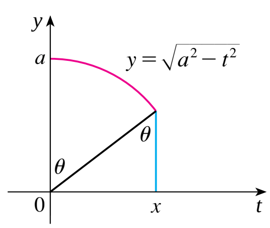
37. Find the volume of the solid obtained by rotating about the x-axis the region enclosed by the curves $y = 9/(x^2 + 9)$, $y = 0$, $x = 0$, and $x = 3$.

38. Find the volume of the solid obtained by rotating about the line $x = 1$ the region under the curve $y = x\sqrt{1 - x^2}$, $0 \leqslant x \leqslant 1$.

39. (a) Use trigonometric substitution to verify that

$$\int_0^x \sqrt{a^2 - t^2}\, dt = \tfrac{1}{2}a^2 \sin^{-1}(x/a) + \tfrac{1}{2}x\sqrt{a^2 - x^2}$$

(b) Use the figure to give trigonometric interpretations of both terms on the right side of the equation in part (a).



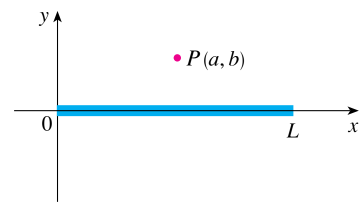
40. The parabola $y = \frac{1}{2}x^2$ divides the disk $x^2 + y^2 \leqslant 8$ into two parts. Find the areas of both parts.

41. A torus is generated by rotating the circle $x^2 + (y - R)^2 = r^2$ about the x-axis. Find the volume enclosed by the torus.

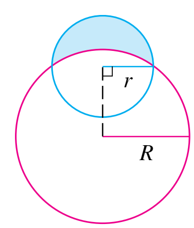
42. A charged rod of length L produces an electric field at point $P(a, b)$ given by

$$E(P) = \int_{-a}^{L-a} \frac{\lambda b}{4\pi\varepsilon_0(x^2 + b^2)^{3/2}}\, dx$$

where λ is the charge density per unit length on the rod and ε_0 is the free space permittivity (see the figure). Evaluate the integral to determine an expression for the electric field $E(P)$.



43. Find the area of the crescent-shaped region (called a *lune*) bounded by arcs of circles with radii r and R. (See the figure.)



44. A water storage tank has the shape of a cylinder with diameter 10 ft. It is mounted so that the circular cross-sections are vertical. If the depth of the water is 7 ft, what percentage of the total capacity is being used?

7.4 Integration of Rational Functions by Partial Fractions

In this section we show how to integrate any rational function (a ratio of polynomials) by expressing it as a sum of simpler fractions, called *partial fractions*, that we already know how to integrate. To illustrate the method, observe that by taking the fractions $2/(x - 1)$ and $1/(x + 2)$ to a common denominator we obtain

$$\frac{2}{x - 1} - \frac{1}{x + 2} = \frac{2(x + 2) - (x - 1)}{(x - 1)(x + 2)} = \frac{x + 5}{x^2 + x - 2}$$

If we now reverse the procedure, we see how to integrate the function on the right side of