could be evaluated by the method of Case III, it's much easier to observe that if $u = x(x^2 + 3) = x^3 + 3x$, then $du = (3x^2 + 3)\,dx$ and so

$$\int \frac{x^2 + 1}{x(x^2 + 3)}\,dx = \tfrac{1}{3}\ln|x^3 + 3x| + C$$

## Rationalizing Substitutions

Some nonrational functions can be changed into rational functions by means of appropriate substitutions. In particular, when an integrand contains an expression of the form $\sqrt[n]{g(x)}$, then the substitution $u = \sqrt[n]{g(x)}$ may be effective. Other instances appear in the exercises.

**EXAMPLE 9** Evaluate $\displaystyle\int \frac{\sqrt{x + 4}}{x}\,dx$.

SOLUTION Let $u = \sqrt{x + 4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $dx = 2u\,du$. Therefore

$$\int \frac{\sqrt{x + 4}}{x}\,dx = \int \frac{u}{u^2 - 4}\,2u\,du = 2\int \frac{u^2}{u^2 - 4}\,du$$

$$= 2\int \left(1 + \frac{4}{u^2 - 4}\right)du$$

We can evaluate this integral either by factoring $u^2 - 4$ as $(u - 2)(u + 2)$ and using partial fractions or by using Formula 6 with $a = 2$:

$$\int \frac{\sqrt{x + 4}}{x}\,dx = 2\int du + 8\int \frac{du}{u^2 - 4} = 2u + 8 \cdot \frac{1}{2 \cdot 2}\ln\left|\frac{u - 2}{u + 2}\right| + C$$

$$= 2\sqrt{x + 4} + 2\ln\left|\frac{\sqrt{x + 4} - 2}{\sqrt{x + 4} + 2}\right| + C$$

## 7.4 Exercises

**1–6** Write out the form of the partial fraction decomposition of the function (as in Example 7). Do not determine the numerical values of the coefficients.

**1.** (a) $\dfrac{1 + 6x}{(4x - 3)(2x + 5)}$ (b) $\dfrac{10}{5x^2 - 2x^3}$

**2.** (a) $\dfrac{x}{x^2 + x - 2}$ (b) $\dfrac{x^2}{x^2 + x + 2}$

**3.** (a) $\dfrac{x^4 + 1}{x^5 + 4x^3}$ (b) $\dfrac{1}{(x^2 - 9)^2}$

**4.** (a) $\dfrac{x^4 - 2x^3 + x^2 + 2x - 1}{x^2 - 2x + 1}$ (b) $\dfrac{x^2 - 1}{x^3 + x^2 + x}$

**5.** (a) $\dfrac{x^6}{x^2 - 4}$ (b) $\dfrac{x^4}{(x^2 - x + 1)(x^2 + 2)^2}$

**6.** (a) $\dfrac{t^6 + 1}{t^6 + t^3}$ (b) $\dfrac{x^5 + 1}{(x^2 - x)(x^4 + 2x^2 + 1)}$

**7–38** Evaluate the integral.

**7.** $\displaystyle\int \frac{x^4}{x - 1}\,dx$

**8.** $\displaystyle\int \frac{3t - 2}{t + 1}\,dt$

**9.** $\displaystyle\int \frac{5x + 1}{(2x + 1)(x - 1)}\,dx$

**10.** $\displaystyle\int \frac{y}{(y + 4)(2y - 1)}\,dy$

**11.** $\displaystyle\int_0^1 \frac{2}{2x^2+3x+1}\,dx$

**12.** $\displaystyle\int_0^1 \frac{x-4}{x^2-5x+6}\,dx$

**13.** $\displaystyle\int \frac{ax}{x^2-bx}\,dx$

**14.** $\displaystyle\int \frac{1}{(x+a)(x+b)}\,dx$

**15.** $\displaystyle\int_3^4 \frac{x^3-2x^2-4}{x^3-2x^2}\,dx$

**16.** $\displaystyle\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx$

**17.** $\displaystyle\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy$

**18.** $\displaystyle\int \frac{x^2+2x-1}{x^3-x}\,dx$

**19.** $\displaystyle\int \frac{x^2+1}{(x-3)(x-2)^2}\,dx$

**20.** $\displaystyle\int \frac{x^2-5x+16}{(2x+1)(x-2)^2}\,dx$

**21.** $\displaystyle\int \frac{x^3+4}{x^2+4}\,dx$

**22.** $\displaystyle\int \frac{ds}{s^2(s-1)^2}$

**23.** $\displaystyle\int \frac{10}{(x-1)(x^2+9)}\,dx$

**24.** $\displaystyle\int \frac{x^2-x+6}{x^3+3x}\,dx$

**25.** $\displaystyle\int \frac{4x}{x^3+x^2+x+1}\,dx$

**26.** $\displaystyle\int \frac{x^2+x+1}{(x^2+1)^2}\,dx$

**27.** $\displaystyle\int \frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)}\,dx$

**28.** $\displaystyle\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}\,dx$

**29.** $\displaystyle\int \frac{x+4}{x^2+2x+5}\,dx$

**30.** $\displaystyle\int \frac{3x^2+x+4}{x^4+3x^2+2}\,dx$

**31.** $\displaystyle\int \frac{1}{x^3-1}\,dx$

**32.** $\displaystyle\int_0^1 \frac{x}{x^2+4x+13}\,dx$

**33.** $\displaystyle\int_0^1 \frac{x^3+2x}{x^4+4x^2+3}\,dx$

**34.** $\displaystyle\int \frac{x^5+x-1}{x^3+1}\,dx$

**35.** $\displaystyle\int \frac{dx}{x(x^2+4)^2}$

**36.** $\displaystyle\int \frac{x^4+3x^2+1}{x^5+5x^3+5x}\,dx$

**37.** $\displaystyle\int \frac{x^2-3x+7}{(x^2-4x+6)^2}\,dx$

**38.** $\displaystyle\int \frac{x^3+2x^2+3x-2}{(x^2+2x+2)^2}\,dx$

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**39–52** Make a substitution to express the integrand as a rational function and then evaluate the integral.

**39.** $\displaystyle\int \frac{\sqrt{x+1}}{x}\,dx$

**40.** $\displaystyle\int \frac{dx}{2\sqrt{x+3}+x}$

**41.** $\displaystyle\int \frac{dx}{x^2+x\sqrt{x}}$

**42.** $\displaystyle\int_0^1 \frac{1}{1+\sqrt[3]{x}}\,dx$

**43.** $\displaystyle\int \frac{x^3}{\sqrt[3]{x^2+1}}\,dx$

**44.** $\displaystyle\int_{1/3}^3 \frac{\sqrt{x}}{x^2+x}\,dx$

**45.** $\displaystyle\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$ [*Hint:* Substitute $u=\sqrt[6]{x}$.]

**46.** $\displaystyle\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx$

**47.** $\displaystyle\int \frac{e^{2x}}{e^{2x}+3e^x+2}\,dx$

**48.** $\displaystyle\int \frac{\sin x}{\cos^2x-3\cos x}\,dx$

**49.** $\displaystyle\int \frac{\sec^2t}{\tan^2t+3\tan t+2}\,dt$

**50.** $\displaystyle\int \frac{e^x}{(e^x-2)(e^{2x}+1)}\,dx$

**51.** $\displaystyle\int \frac{dx}{1+e^x}$

**52.** $\displaystyle\int \frac{\cosh t}{\sinh^2t+\sinh^4t}\,dt$

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**53–54** Use integration by parts, together with the techniques of this section, to evaluate the integral.

**53.** $\displaystyle\int \ln(x^2-x+2)\,dx$

**54.** $\displaystyle\int x\tan^{-1}x\,dx$

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**55.** Use a graph of $f(x)=1/(x^2-2x-3)$ to decide whether $\int_0^2 f(x)\,dx$ is positive or negative. Use the graph to give a rough estimate of the value of the integral and then use partial fractions to find the exact value.

**56.** Evaluate

$$\int \frac{1}{x^2+k}\,dx$$

by considering several cases for the constant $k$.

**57–58** Evaluate the integral by completing the square and using Formula 6.

**57.** $\displaystyle\int \frac{dx}{x^2-2x}$

**58.** $\displaystyle\int \frac{2x+1}{4x^2+12x-7}\,dx$

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**59.** The German mathematician Karl Weierstrass (1815–1897) noticed that the substitution $t=\tan(x/2)$ will convert any rational function of $\sin x$ and $\cos x$ into an ordinary rational function of $t$.

(a) If $t=\tan(x/2)$, $-\pi<x<\pi$, sketch a right triangle or use trigonometric identities to show that

$$\cos\left(\frac{x}{2}\right)=\frac{1}{\sqrt{1+t^2}} \quad\text{and}\quad \sin\left(\frac{x}{2}\right)=\frac{t}{\sqrt{1+t^2}}$$

(b) Show that

$$\cos x=\frac{1-t^2}{1+t^2} \quad\text{and}\quad \sin x=\frac{2t}{1+t^2}$$

(c) Show that

$$dx=\frac{2}{1+t^2}\,dt$$

**60–63** Use the substitution in Exercise 59 to transform the integrand into a rational function of $t$ and then evaluate the integral.

**60.** $\displaystyle\int \frac{dx}{1-\cos x}$

**61.** $\displaystyle\int \frac{1}{3\sin x-4\cos x}\,dx$

**62.** $\displaystyle\int_{\pi/3}^{\pi/2} \frac{1}{1+\sin x-\cos x}\,dx$

**63.** $\displaystyle\int_0^{\pi/2} \frac{\sin 2x}{2 + \cos x}\,dx$

**64–65** Find the area of the region under the given curve from 1 to 2.

**64.** $y = \dfrac{1}{x^3 + x}$

**65.** $y = \dfrac{x^2 + 1}{3x - x^2}$

**66.** Find the volume of the resulting solid if the region under the curve $y = 1/(x^2 + 3x + 2)$ from $x = 0$ to $x = 1$ is rotated about (a) the $x$-axis and (b) the $y$-axis.

**67.** One method of slowing the growth of an insect population without using pesticides is to introduce into the population a number of sterile males that mate with fertile females but produce no offspring. If $P$ represents the number of female insects in a population, $S$ the number of sterile males introduced each generation, and $r$ the population's natural growth rate, then the female population is related to time $t$ by

$$t = \int \frac{P + S}{P[(r - 1)P - S]}\,dP$$

Suppose an insect population with 10,000 females grows at a rate of $r = 0.10$ and 900 sterile males are added. Evaluate the integral to give an equation relating the female population to time. (Note that the resulting equation can't be solved explicitly for $P$.)

**68.** Factor $x^4 + 1$ as a difference of squares by first adding and subtracting the same quantity. Use this factorization to evaluate $\int 1/(x^4 + 1)\,dx$.

CAS **69.** (a) Use a computer algebra system to find the partial fraction decomposition of the function

$$f(x) = \frac{4x^3 - 27x^2 + 5x - 32}{30x^5 - 13x^4 + 50x^3 - 286x^2 - 299x - 70}$$

(b) Use part (a) to find $\int f(x)\,dx$ (by hand) and compare with the result of using the CAS to integrate $f$ directly. Comment on any discrepancy.

CAS **70.** (a) Find the partial fraction decomposition of the function

$$f(x) = \frac{12x^5 - 7x^3 - 13x^2 + 8}{100x^6 - 80x^5 + 116x^4 - 80x^3 + 41x^2 - 20x + 4}$$

(b) Use part (a) to find $\int f(x)\,dx$ and graph $f$ and its indefinite integral on the same screen.

(c) Use the graph of $f$ to discover the main features of the graph of $\int f(x)\,dx$.

**71.** Suppose that $F$, $G$, and $Q$ are polynomials and

$$\frac{F(x)}{Q(x)} = \frac{G(x)}{Q(x)}$$

for all $x$ except when $Q(x) = 0$. Prove that $F(x) = G(x)$ for all $x$. [*Hint:* Use continuity.]

**72.** If $f$ is a quadratic function such that $f(0) = 1$ and

$$\int \frac{f(x)}{x^2(x + 1)^3}\,dx$$

is a rational function, find the value of $f'(0)$.

**73.** If $a \neq 0$ and $n$ is a positive integer, find the partial fraction decomposition of

$$f(x) = \frac{1}{x^n(x - a)}$$

*Hint:* First find the coefficient of $1/(x - a)$. Then subtract the resulting term and simplify what is left.

## 7.5 Strategy for Integration

As we have seen, integration is more challenging than differentiation. In finding the derivative of a function it is obvious which differentiation formula we should apply. But it may not be obvious which technique we should use to integrate a given function.

Until now individual techniques have been applied in each section. For instance, we usually used substitution in Exercises 5.5, integration by parts in Exercises 7.1, and partial fractions in Exercises 7.4. But in this section we present a collection of miscellaneous integrals in random order and the main challenge is to recognize which technique or formula to use. No hard and fast rules can be given as to which method applies in a given situation, but we give some advice on strategy that you may find useful.

A prerequisite for applying a strategy is a knowledge of the basic integration formulas. In the following table we have collected the integrals from our previous list together with several additional formulas that we have learned in this chapter. Most of them should be memorized. It is useful to know them all, but the ones marked with an asterisk need not be