7.8 Exercises

1. Explain why each of the following integrals is improper.

(a) $\int_1^2 \frac{x}{x-1}\,dx$ (b) $\int_0^\infty \frac{1}{1+x^3}\,dx$

(c) $\int_{-\infty}^\infty x^2 e^{-x^2}\,dx$ (d) $\int_0^{\pi/4} \cot x\,dx$

2. Which of the following integrals are improper? Why?

(a) $\int_0^{\pi/4} \tan x\,dx$ (b) $\int_0^{\pi} \tan x\,dx$

(c) $\int_{-1}^1 \frac{dx}{x^2 - x - 2}$ (d) $\int_0^\infty e^{-x^3}\,dx$

3. Find the area under the curve $y = 1/x^3$ from $x = 1$ to $x = t$ and evaluate it for $t = 10$, 100, and 1000. Then find the total area under this curve for $x \geqslant 1$.

4. (a) Graph the functions $f(x) = 1/x^{1.1}$ and $g(x) = 1/x^{0.9}$ in the viewing rectangles $[0, 10]$ by $[0, 1]$ and $[0, 100]$ by $[0, 1]$.
(b) Find the areas under the graphs of f and g from $x = 1$ to $x = t$ and evaluate for $t = 10$, 100, 10^4, 10^6, 10^{10}, and 10^{20}.
(c) Find the total area under each curve for $x \geqslant 1$, if it exists.

5–40 Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

5. $\int_3^\infty \frac{1}{(x-2)^{3/2}}\,dx$

6. $\int_0^\infty \frac{1}{\sqrt[4]{1+x}}\,dx$

7. $\int_{-\infty}^0 \frac{1}{3-4x}\,dx$

8. $\int_1^\infty \frac{1}{(2x+1)^3}\,dx$

9. $\int_2^\infty e^{-5p}\,dp$

10. $\int_{-\infty}^0 2^r\,dr$

11. $\int_0^\infty \frac{x^2}{\sqrt{1+x^3}}\,dx$

12. $\int_{-\infty}^\infty (y^3 - 3y^2)\,dy$

13. $\int_{-\infty}^\infty x e^{-x^2}\,dx$

14. $\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

15. $\int_0^\infty \sin^2\alpha\,d\alpha$

16. $\int_{-\infty}^\infty \cos \pi t\,dt$

17. $\int_1^\infty \frac{1}{x^2+x}\,dx$

18. $\int_2^\infty \frac{dv}{v^2+2v-3}$

19. $\int_{-\infty}^0 z e^{2z}\,dz$

20. $\int_2^\infty y e^{-3y}\,dy$

21. $\int_1^\infty \frac{\ln x}{x}\,dx$

22. $\int_{-\infty}^\infty x^3 e^{-x^4}\,dx$

23. $\int_{-\infty}^\infty \frac{x^2}{9+x^6}\,dx$

24. $\int_0^\infty \frac{e^x}{e^{2x}+3}\,dx$

25. $\int_e^\infty \frac{1}{x(\ln x)^3}\,dx$

26. $\int_0^\infty \frac{x \arctan x}{(1+x^2)^2}\,dx$

27. $\int_0^1 \frac{3}{x^5}\,dx$

28. $\int_2^3 \frac{1}{\sqrt{3-x}}\,dx$

29. $\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}}$

30. $\int_6^8 \frac{4}{(x-6)^3}\,dx$

31. $\int_{-2}^3 \frac{1}{x^4}\,dx$

32. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

33. $\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx$

34. $\int_0^5 \frac{w}{w-2}\,dw$

35. $\int_0^3 \frac{dx}{x^2-6x+5}$

36. $\int_{\pi/2}^{\pi} \csc x\,dx$

37. $\int_{-1}^0 \frac{e^{1/x}}{x^3}\,dx$

38. $\int_0^1 \frac{e^{1/x}}{x^3}\,dx$

39. $\int_0^2 z^2 \ln z\,dz$

40. $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$

41–46 Sketch the region and find its area (if the area is finite).

41. $S = \{(x, y) \mid x \geqslant 1,\ 0 \leqslant y \leqslant e^{-x}\}$

42. $S = \{(x, y) \mid x \leqslant 0,\ 0 \leqslant y \leqslant e^{x}\}$

43. $S = \{(x, y) \mid x \geqslant 1,\ 0 \leqslant y \leqslant 1/(x^3 + x)\}$

44. $S = \{(x, y) \mid x \geqslant 0,\ 0 \leqslant y \leqslant x e^{-x}\}$

45. $S = \{(x, y) \mid 0 \leqslant x < \pi/2,\ 0 \leqslant y \leqslant \sec^2 x\}$

46. $S = \{(x, y) \mid -2 < x \leqslant 0,\ 0 \leqslant y \leqslant 1/\sqrt{x+2}\}$

47. (a) If $g(x) = (\sin^2 x)/x^2$, use your calculator or computer to make a table of approximate values of $\int_1^t g(x)\,dx$ for $t = 2$, 5, 10, 100, 1000, and 10,000. Does it appear that $\int_1^\infty g(x)\,dx$ is convergent?
(b) Use the Comparison Theorem with $f(x) = 1/x^2$ to show that $\int_1^\infty g(x)\,dx$ is convergent.
(c) Illustrate part (b) by graphing f and g on the same screen for $1 \leqslant x \leqslant 10$. Use your graph to explain intuitively why $\int_1^\infty g(x)\,dx$ is convergent.

48. (a) If $g(x) = 1/(\sqrt{x} - 1)$, use your calculator or computer to make a table of approximate values of $\int_2^t g(x)\,dx$ for $t = 5$, 10, 100, 1000, and 10,000. Does it appear that $\int_2^\infty g(x)\,dx$ is convergent or divergent?

(b) Use the Comparison Theorem with $f(x) = 1/\sqrt{x}$ to show that $\int_2^\infty g(x)\,dx$ is divergent.

(c) Illustrate part (b) by graphing f and g on the same screen for $2 \leqslant x \leqslant 20$. Use your graph to explain intuitively why $\int_2^\infty g(x)\,dx$ is divergent.

49–54 Use the Comparison Theorem to determine whether the integral is convergent or divergent.

49. $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx$

50. $\displaystyle\int_1^\infty \frac{2+e^{-x}}{x}\,dx$

51. $\displaystyle\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx$

52. $\displaystyle\int_0^\infty \frac{\arctan x}{2+e^x}\,dx$

53. $\displaystyle\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\,dx$

54. $\displaystyle\int_0^\pi \frac{\sin^2 x}{\sqrt{x}}\,dx$

55. The integral

$$\int_0^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx$$

is improper for two reasons: The interval $[0, \infty)$ is infinite and the integrand has an infinite discontinuity at 0. Evaluate it by expressing it as a sum of improper integrals of Type 2 and Type 1 as follows:

$$\int_0^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx = \int_0^1 \frac{1}{\sqrt{x}\,(1+x)}\,dx + \int_1^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx$$

56. Evaluate

$$\int_2^\infty \frac{1}{x\sqrt{x^2-4}}\,dx$$

by the same method as in Exercise 55.

57–59 Find the values of p for which the integral converges and evaluate the integral for those values of p.

57. $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$

58. $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^p}\,dx$

59. $\displaystyle\int_0^1 x^p \ln x\,dx$

60. (a) Evaluate the integral $\int_0^\infty x^n e^{-x}\,dx$ for $n = 0, 1, 2$, and 3.

(b) Guess the value of $\int_0^\infty x^n e^{-x}\,dx$ when n is an arbitrary positive integer.

(c) Prove your guess using mathematical induction.

61. (a) Show that $\int_{-\infty}^\infty x\,dx$ is divergent.

(b) Show that

$$\lim_{t\to\infty} \int_{-t}^t x\,dx = 0$$

This shows that we can't define

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{t\to\infty} \int_{-t}^t f(x)\,dx$$

62. The *average speed* of molecules in an ideal gas is

$$\bar{v} = \frac{4}{\sqrt{\pi}}\left(\frac{M}{2RT}\right)^{3/2} \int_0^\infty v^3 e^{-Mv^2/(2RT)}\,dv$$

where M is the molecular weight of the gas, R is the gas constant, T is the gas temperature, and v is the molecular speed. Show that

$$\bar{v} = \sqrt{\frac{8RT}{\pi M}}$$

63. We know from Example 1 that the region $\mathscr{R} = \{(x, y) \mid x \geqslant 1,\ 0 \leqslant y \leqslant 1/x\}$ has infinite area. Show that by rotating $\mathscr{R}$ about the x-axis we obtain a solid with finite volume.

64. Use the information and data in Exercise 29 of Section 6.4 to find the work required to propel a 1000-kg space vehicle out of the earth's gravitational field.

65. Find the *escape velocity* v_0 that is needed to propel a rocket of mass m out of the gravitational field of a planet with mass M and radius R. Use Newton's Law of Gravitation (see Exercise 29 in Section 6.4) and the fact that the initial kinetic energy of $\frac{1}{2}mv_0^2$ supplies the needed work.

66. Astronomers use a technique called *stellar stereography* to determine the density of stars in a star cluster from the observed (two-dimensional) density that can be analyzed from a photograph. Suppose that in a spherical cluster of radius R the density of stars depends only on the distance r from the center of the cluster. If the perceived star density is given by $y(s)$, where s is the observed planar distance from the center of the cluster, and $x(r)$ is the actual density, it can be shown that

$$y(s) = \int_s^R \frac{2r}{\sqrt{r^2-s^2}}\,x(r)\,dr$$

If the actual density of stars in a cluster is $x(r) = \frac{1}{2}(R-r)^2$, find the perceived density $y(s)$.

67. A manufacturer of lightbulbs wants to produce bulbs that last about 700 hours but, of course, some bulbs burn out faster than others. Let $F(t)$ be the fraction of the company's bulbs that burn out before t hours, so $F(t)$ always lies between 0 and 1.

(a) Make a rough sketch of what you think the graph of F might look like.

(b) What is the meaning of the derivative $r(t) = F'(t)$?

(c) What is the value of $\int_0^\infty r(t)\,dt$? Why?

68. As we saw in Section 3.8, a radioactive substance decays exponentially: The mass at time t is $m(t) = m(0)e^{kt}$, where $m(0)$ is the initial mass and k is a negative constant. The *mean life* M of an atom in the substance is

$$M = -k\int_0^\infty te^{kt}\,dt$$

For the radioactive carbon isotope, ^{14}C, used in radiocarbon dating, the value of k is -0.000121. Find the mean life of a ^{14}C atom.

69. Determine how large the number a has to be so that

$$\int_a^\infty \frac{1}{x^2+1}\,dx < 0.001$$

70. Estimate the numerical value of $\int_0^\infty e^{-x^2}\,dx$ by writing it as the sum of $\int_0^4 e^{-x^2}\,dx$ and $\int_4^\infty e^{-x^2}\,dx$. Approximate the first integral by using Simpson's Rule with $n = 8$ and show that the second integral is smaller than $\int_4^\infty e^{-4x}\,dx$, which is less than 0.0000001.

71. If $f(t)$ is continuous for $t \geqslant 0$, the *Laplace transform* of f is the function F defined by

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

and the domain of F is the set consisting of all numbers s for which the integral converges. Find the Laplace transforms of the following functions.
(a) $f(t) = 1$ (b) $f(t) = e^t$ (c) $f(t) = t$

72. Show that if $0 \leqslant f(t) \leqslant Me^{at}$ for $t \geqslant 0$, where M and a are constants, then the Laplace transform $F(s)$ exists for $s > a$.

73. Suppose that $0 \leqslant f(t) \leqslant Me^{at}$ and $0 \leqslant f'(t) \leqslant Ke^{at}$ for $t \geqslant 0$, where f' is continuous. If the Laplace transform of $f(t)$ is $F(s)$ and the Laplace transform of $f'(t)$ is $G(s)$, show that

$$G(s) = sF(s) - f(0) \qquad s > a$$

74. If $\int_{-\infty}^\infty f(x)\,dx$ is convergent and a and b are real numbers, show that

$$\int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx = \int_{-\infty}^b f(x)\,dx + \int_b^\infty f(x)\,dx$$

75. Show that $\int_0^\infty x^2e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty e^{-x^2}\,dx$.

76. Show that $\int_0^\infty e^{-x^2}\,dx = \int_0^1 \sqrt{-\ln y}\,dy$ by interpreting the integrals as areas.

77. Find the value of the constant C for which the integral

$$\int_0^\infty \left(\frac{1}{\sqrt{x^2+4}} - \frac{C}{x+2}\right) dx$$

converges. Evaluate the integral for this value of C.

78. Find the value of the constant C for which the integral

$$\int_0^\infty \left(\frac{x}{x^2+1} - \frac{C}{3x+1}\right) dx$$

converges. Evaluate the integral for this value of C.

79. Suppose f is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 1$. Is it possible that $\int_0^\infty f(x)\,dx$ is convergent?

80. Show that if $a > -1$ and $b > a + 1$, then the following integral is convergent.

$$\int_0^\infty \frac{x^a}{1+x^b}\,dx$$

7 Review

Concept Check

1. State the rule for integration by parts. In practice, how do you use it?

2. How do you evaluate $\int \sin^m x \cos^n x\,dx$ if m is odd? What if n is odd? What if m and n are both even?

3. If the expression $\sqrt{a^2 - x^2}$ occurs in an integral, what substitution might you try? What if $\sqrt{a^2 + x^2}$ occurs? What if $\sqrt{x^2 - a^2}$ occurs?

4. What is the form of the partial fraction decomposition of a rational function $P(x)/Q(x)$ if the degree of P is less than the degree of Q and $Q(x)$ has only distinct linear factors? What if a linear factor is repeated? What if $Q(x)$ has an irreducible quadratic factor (not repeated)? What if the quadratic factor is repeated?

5. State the rules for approximating the definite integral $\int_a^b f(x)\,dx$ with the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule. Which would you expect to give the best estimate? How do you approximate the error for each rule?

6. Define the following improper integrals.
(a) $\int_a^\infty f(x)\,dx$ (b) $\int_{-\infty}^b f(x)\,dx$ (c) $\int_{-\infty}^\infty f(x)\,dx$

7. Define the improper integral $\int_a^b f(x)\,dx$ for each of the following cases.
(a) f has an infinite discontinuity at a.
(b) f has an infinite discontinuity at b.
(c) f has an infinite discontinuity at c, where $a < c < b$.

8. State the Comparison Theorem for improper integrals.