This shows that the expressions in the parentheses must be the Fourier coefficients b_n of $f(x)$; that is, by (4) in Sec. 11.3,

$$b_n = A_n^* \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} \, dx.$$

From this and (16) we see that the solution of our problem is

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a} \tag{17}$$

where

$$A_n^* = \frac{2}{a \sinh (n\pi b/a)} \int_0^a f(x) \sin \frac{n\pi x}{a} \, dx. \tag{18}$$

We have obtained this solution formally, neither considering convergence nor showing that the series for u, u_{xx}, and u_{yy} have the right sums. This can be proved if one assumes that f and f' are continuous and f'' is piecewise continuous on the interval $0 \leqq x \leqq a$. The proof is somewhat involved and relies on uniform convergence. It can be found in [C4] listed in App. 1.

Unifying Power of Methods. Electrostatics, Elasticity

The Laplace equation (14) also governs the electrostatic potential of electrical charges in any region that is free of these charges. Thus our steady-state heat problem can also be interpreted as an electrostatic potential problem. Then (17), (18) is the potential in the rectangle R when the upper side of R is at potential $f(x)$ and the other three sides are grounded.

Actually, in the steady-state case, the two-dimensional wave equation (to be considered in Secs. 12.7, 12.8) also reduces to (14). Then (17), (18) is the displacement of a rectangular elastic membrane (rubber sheet, drumhead) that is fixed along its boundary, with three sides lying in the xy-plane and the fourth side given the displacement $f(x)$.

This is another impressive demonstration of the ***unifying power*** of mathematics. It illustrates that ***entirely different physical systems may have the same mathematical model*** and can thus be treated by the same mathematical methods.

PROBLEM SET 12.5

1. **WRITING PROJECT. Wave and Heat Equations.** Compare the two PDEs with respect to type, general behavior of eigenfunctions, and kind of boundary and initial conditions and resulting practical problems. Also discuss the difference between Figs. 288 in Sec. 12.3 and 292.
2. **(Eigenfunctions)** Sketch (or graph) and compare the first three eigenfunctions (8) with $B_n = 1$, $c = 1$, $L = \pi$ for $t = 0, 0.2, 0.4, 0.6, 0.8, 1.0$.
3. **(Decay)** How does the rate of decay of (8) with fixed n depend on the specific heat, the density, and the thermal conductivity of the material?

4. If the first eigenfunction (8) of the bar decreases to half its value within 10 sec, what is the value of the diffusivity?

5–9 LATERALLY INSULATED BAR

A laterally insulated bar of length 10 cm and constant cross-sectional area 1 cm^2, of density 10.6 gm/cm^3, thermal conductivity 1.04 cal/(cm sec °C), and specific heat 0.056 cal/(gm °C) (this corresponds to silver, a good heat conductor) has initial temperature $f(x)$ and is kept at 0°C at the ends $x = 0$ and $x = 10$. Find the temperature $u(x, t)$ at later times. Here, $f(x)$ equals:

5. $f(x) = \sin 0.4\pi x$
6. $f(x) = \sin 0.1\pi x + \frac{1}{2}\sin 0.2\pi x$
7. $f(x) = 0.2x$ if $0 < x < 5$ and 0 otherwise
8. $f(x) = 1 - 0.2|x - 5|$
9. $f(x) = x$ if $0 < x < 2.5$, $f(x) = 2.5$ if $2.5 < x < 7.5$, $f(x) = 10 - x$ if $7.5 < x < 10$
10. **(Arbitrary temperatures at ends)** If the ends $x = 0$ and $x = L$ of the bar in the text are kept at constant temperatures U_1 and U_2, respectively, what is the temperature $u_I(x)$ in the bar after a long time (theoretically, as $t \to \infty$)? First guess, then calculate.
11. In Prob. 10 find the temperature at any time.
12. **(Changing end temperatures)** Assume that the ends of the bar in Probs. 5–9 have been kept at 100°C for a long time. Then at some instant, call it $t = 0$, the temperature at $x = L$ is suddenly changed to 0°C and kept at 0°C, whereas the temperature at $x = 0$ is kept at 100°C. Find the temperature in the middle of the bar at $t = 1, 2, 3, 10, 50$ sec. First guess, then calculate.

BAR UNDER ADIABATIC CONDITIONS

"Adiabatic" means no heat exchange with the neighborhood, because the bar is completely insulated, also at the ends. *Physical Information:* The heat flux at the ends is proportional to the value of $\partial u/\partial x$ there.

13. Show that for the completely insulated bar, $u_x(0, t) = 0$, $u_x(L, t) = 0$, $u(x, t) = f(x)$ and separation of variables gives the following solution, with A_n given by (2) in Sec. 11.3.

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} e^{-(cn\pi/L)^2 t}$$

14–19 Find the temperature in Prob. 13 with $L = \pi$, $c = 1$, and

14. $f(x) = x$
15. $f(x) = 1$
16. $f(x) = 0.5 \cos 4x$
17. $f(x) = \pi^2 - x^2$
18. $f(x) = \frac{1}{2}\pi - |x - \frac{1}{2}\pi|$
19. $f(x) = (x - \frac{1}{2}\pi)^2$
20. Find the temperature of the bar in Prob. 13 if the left end is kept at 0°C, the right end is insulated, and the initial temperature is $U_0 = const.$
21. The **boundary condition of heat transfer**

$$(19) \qquad -u_x(\pi, t) = k[u(\pi, t) - u_0]$$

applies when a bar of length π with $c = 1$ is laterally insulated, the left end $x = 0$ is kept at 0°C, and at the right end heat is flowing into air of constant temperature u_0. Let $k = 1$ for simplicity, and $u_0 = 0$. Show that a solution is $u(x, t) = \sin px\, e^{-p^2 t}$, where p is a solution of $\tan p\pi = -p$. Show graphically that this equation has infinitely many positive solutions $p_1, p_2, p_3, \cdots$, where $p_n > n - \frac{1}{2}$ and $\lim_{n\to\infty} (p_n - n + \frac{1}{2}) = 0$. (Formula (19) is also known as **radiation boundary condition,** but this is misleading; see Ref. [C3], p. 19.)

22. **(Discontinuous f)** Solve (1), (2), (3) with $L = \pi$ and $f(x) = U_0 = const$ $(\neq 0)$ if $0 < x < \pi/2$, $f(x) = 0$ if $\pi/2 < x < \pi$.
23. **(Heat flux)** The *heat flux* of a solution $u(x, t)$ across $x = 0$ is defined by $\phi(t) = -Ku_x(0, t)$. Find $\phi(t)$ for the solution (9). Explain the name. Is it physically understandable that ϕ goes to 0 as $t \to \infty$?

OTHER HEAT EQUATIONS

24. **(Bar with heat generation)** If heat is generated at a constant rate throughout a bar of length $L = \pi$ with initial temperature $f(x)$ and the ends at $x = 0$ and π are kept at temperature 0, the heat equation is $u_t = c^2 u_{xx} + H$ with constant $H > 0$. Solve this problem. *Hint.* Set $u = v - Hx(x - \pi)/(2c^2)$.
25. **(Convection)** If heat in the bar in the text is free to flow through an end into the surrounding medium kept at 0°C, the PDE becomes $v_t = c^2 v_{xx} - \beta v$. Show that it can be reduced to the form (1) by setting $v(x, t) = u(x, t)w(t)$.
26. Consider $v_t = c^2 v_{xx} - v \quad (0 < x < L,\ t > 0)$, $v(0, t) = 0$, $v(L, t) = 0$, $v(x, 0) = f(x)$, where the term $-v$ models heat transfer to the surrounding medium kept at temperature 0. Reduce this PDE by setting $v(x, t) = u(x, t)w(t)$ with w such that u is given by (9), (10).
27. **(Nonhomogeneous heat equation)** Show that the problem modeled by

$$u_t - c^2 u_{xx} = Ne^{-\alpha x}$$

and (2), (3) can be reduced to a problem for the homogeneous heat equation by setting

$$u(x, t) = v(x, t) + w(x)$$

and determining w so that v satisfies the homogeneous PDE and the conditions $v(0, t) = v(L, t) = 0$, $v(x, 0) = f(x) - w(x)$. (The term $Ne^{-\alpha x}$ may represent heat loss due to radioactive decay in the bar.)

28–35 TWO-DIMENSIONAL PROBLEMS

28. **(Laplace equation)** Find the potential in the rectangle $0 \leqq x \leqq 20$, $0 \leqq y \leqq 40$ whose upper side is kept at potential 220 V and whose other sides are grounded.

29. Find the potential in the square $0 \leqq x \leqq 2$, $0 \leqq y \leqq 2$ if the upper side is kept at the potential $\sin \frac{1}{2}\pi x$ and the other sides are grounded.

30. **CAS PROJECT. Isotherms.** Find the steady-state solutions (temperatures) in the square plate in Fig. 294 with $a = 2$ satisfying the following boundary conditions. Graph isotherms.

(a) $u = \sin \pi x$ on the upper side, 0 on the others.

(b) $u = 0$ on the vertical sides, assuming that the other sides are perfectly insulated.

(c) Boundary conditions of your choice (such that the solution is not identically zero).

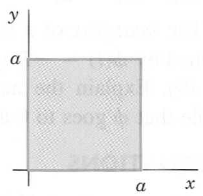

Fig. 294. Square plate

31. **(Heat flow in a plate)** The faces of the thin square plate in Fig. 294 with side $a = 24$ are perfectly insulated. The upper side is kept at 20°C and the other sides are kept at 0°C. Find the steady-state temperature $u(x, y)$ in the plate.

32. Find the steady-state temperature in the plate in Prob. 31 if the lower side is kept at U_0°C, the upper side at U_1°C, and the other sides are kept at 0°C. *Hint:* Split into two problems in which the boundary temperature is 0 on three sides for each problem.

33. **(Mixed boundary value problem)** Find the steady-state temperature in the plate in Prob. 31 with the upper and lower sides perfectly insulated, the left side kept at 0°C, and the right side kept at $f(y)$°C.

34. **(Radiation)** Find steady-state temperatures in the rectangle in Fig. 293 with the upper and left sides perfectly insulated and the right side radiating into a medium at 0°C according to $u_x(a, y) + hu(a, y) = 0$, $h > 0$ constant. (You will get many solutions since no condition on the lower side is given.)

35. Find formulas similar to (17), (18) for the temperature in the rectangle R of the text when the lower side of R is kept at temperature $f(x)$ and the other sides are kept at 0°C.

12.6 Heat Equation: Solution by Fourier Integrals and Transforms

Our discussion of the heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

in the last section extends to bars of infinite length, which are good models of very long bars or wires (such as a wire of length, say, 300 ft). Then the role of Fourier series in the solution process will be taken by **Fourier integrals** (Sec. 11.7).

Let us illustrate the method by solving (1) for a bar that extends to infinity on both sides (and is laterally insulated as before). Then we do not have boundary conditions, but only the **initial condition**

$$u(x, 0) = f(x) \qquad (-\infty < x < \infty) \tag{2}$$

where $f(x)$ is the given initial temperature of the bar.

To solve this problem, we start as in the last section, substituting $u(x, t) = F(x)G(t)$ into (1). This gives the two ODEs

$$F'' + p^2 F = 0 \qquad \text{[see (5), Sec. 12.5]} \tag{3}$$